Method of Undetermined Coefficients

Problem. Find a particular solution y_p of the constant coefficients linear equation

$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = g(x).$$

We assume that

$$g(x) = [\text{polynomial}] \times [\text{exponential}] \times [\text{sinusoid}].$$

More precisely, we assume that

$$g(x) = p(x)\, e^{\alpha x}\,(a\cos(\omega x) + b\sin(\omega x)),$$

where $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m$.

Basic Guess.

- If $\omega = 0$, so that $g(x) = p(x)\, e^{\alpha x}$, we look for y_p in the form

  $$y_p = (A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)\, e^{\alpha x}$$

  for the undetermined coefficients $A_0, A_1, A_2, \ldots, A_m$.

- If $\omega \neq 0$, we look for y_p in the form

  $$y_p = (A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m) e^{\alpha x} \cos(\omega x) + (B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m) e^{\alpha x} \sin(\omega x)$$

  for the undetermined coefficients $A_0, A_1, A_2, \ldots, A_m$ and $B_0, B_1, B_2, \ldots, B_m$.

We always choose m in both cases above to match the degree of the polynomial $p(x)$. If $g(x)$ does not include an exponential term, simply set $\alpha = 0$.

Examples.

$g(x)$	Form of y_p
5	A
$3x^2$	$Ax^2 + Bx + C$
$5x^3 + 2x - 3$	$Ax^3 + Bx^2 + Cx + D$
$3e^{4x}$	Ae^{4x}
$(5x+3)e^{2x}$	$(Ax+B)e^{2x}$
$x^2 e^{6x}$	$(Ax^2 + Bx + C)e^{6x}$
$5\cos(2x)$	$A\cos(2x) + B\sin(2x)$
$e^{-2x}(\cos(3x) + 2\sin(3x))$	$Ae^{-2x}\cos(3x) + Be^{-2x}\sin(3x)$
$6x^2\sin(5x)$	$(Ax^2 + Bx + C)\cos(5x) + (Dx^2 + Ex + F)\sin(5x)$
$xe^{3x}\cos(2x)$	$(Ax+B)e^{3x}\cos(2x) + (Cx+D)e^{3x}\sin(2x)$

If $g(x) = g_1(x) + g_2(x) + \cdots + g_k(x)$, we apply the procedure separately to each $g_i(x)$ to produce a particular solution

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}.$$

Example. If

$$g(x) = x^2 + 4xe^{3x} + 5\cos(2x),$$

then the basic form of y_p is

$$y_p = \underbrace{Ax^2 + Bx + C}_{y_{p_1}} + \underbrace{(Dx + E)e^{3x}}_{y_{p_2}} + \underbrace{F\cos(2x) + G\sin(2x)}_{y_{p_3}}.$$

"Bad Case". If any of the y_{p_i} contains terms that duplicate terms in the homogeneous solution y_c, then that y_{p_i} must be multiplied by x^n, where n is the smallest positive integer that eliminates the duplication.

Example 1. Consider the equation

$$y'' - 6y' + 8y = 3e^{2x}.$$

Since the roots of $m^2 - 6m + 8 = 0$ are $m = 2$ and $m = 4$, then $y_c = c_1e^{2x} + c_2e^{4x}$. Since $g(x) = 3e^{2x}$, the basic form of y_p is $y_p = Ae^{2x}$. Since Ae^{2x} duplicates the term c_1e^{2x} in y_c, we have a "bad case". We should multiply Ae^{2x} by x and use use the following

$$\boxed{y_p = Axe^{2x}}$$

Example 2. Consider the equation

$$y'' - 6y' + 9y = 5xe^{3x}.$$

The only root of $m^2 - 6m + 9 = 0$ is $m = 3$. Then, $y_c = c_1e^{3x} + c_2xe^{3x}$. Since $g(x) = 5xe^{3x}$, the basic form of y_p is $y_p = (Ax + B)e^{3x}$. Since this y_p duplicates terms in y_c, we have a "bad case". Multiplying by x would still involve terms duplicated in y_c. We should then multiply by x^2 and use the following.

$$\boxed{y_p = (Ax^3 + Bx^2)e^{3x}}$$

Example 3. Consider the equation

$$y'' + 9y = 5e^x + 2\sin(3x).$$

The roots of $m^2 + 9 = 0$ are $m = \pm 3i$. Then, $y_c = c_1\cos(3x) + c_2\sin(3x)$. Since $g(x) = g_1(x) + g_2(x)$, where $g_1(x) = 5e^x$ and $g_2(x) = 2\sin(3x)$, then $y_p = y_{p_1} + y_{p_2}$. The basic forms of y_{p_1} and y_{p_2} are

$$y_{p_1} = Ae^x \quad \text{and} \quad y_{p_2} = B\cos(3x) + C\sin(3x).$$

Since y_{p_2} duplicates y_c, we have a "bad case". We should multiply y_{p_2} by x and use use the following .

$$\boxed{y_p = Ae^x + Bx\cos(3x) + Cx\sin(3x)}$$

Observe that we do not multiply y_{p_1} by x.

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