Constant Coefficients and Cauchy-Euler Equations

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**Constant Coefficients.**
A homogeneous second-order constant coefficient equation is of the form

\[ ay''(x) + by'(x) + cy(x) = 0. \]  

(1)

To solve the equation, plug \( y = e^{mx} \) into (1) and deduce the characteristic equation

\[ am^2 + bm + c = 0. \]  

(2)

The general solution depends on the nature of the roots of (2).

<table>
<thead>
<tr>
<th>Roots</th>
<th>General Solution</th>
</tr>
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<tbody>
<tr>
<td>( m_1 ) and ( m_2 ) are distinct real roots</td>
<td>( y = c_1e^{m_1x} + c_2e^{m_2x} )</td>
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<td>( m_{1,2} = \alpha \pm \beta i ) are complex conjugates roots</td>
<td>( y = e^{\alpha x}(c_1 \sin(\beta x) + c_2 \cos(\beta x)) )</td>
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**Cauchy-Euler.**
A homogeneous second-order Cauchy-Euler equation is of the form

\[ ax^2y''(x) + bxy'(x) + cy(x) = 0. \]  

(3)

To solve the equation, plug \( y = x^m \) into (3) and deduce the characteristic equation

\[ am^2 + (b - a)m + c = 0. \]  

(4)

The general solution depends on the nature of the roots of (4).

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**Note.** The solutions given for equation (1) are valid over the whole real line. The solutions given for equation (3) are valid over the interval \((0, \infty)\).
The similarity between the solutions of the constant coefficients and Cauchy-Euler equations can be explained as follows. Start from the Cauchy-Euler equation

$$ax^2y''(x) + bxy'(x) + cy(x) = 0.$$  

(5)

Introduce a new variable $t$ defined by

$$t = \ln x \iff x = e^t.$$  

Observe that the $x$-interval $(0, \infty)$ transforms into the $t$-interval $(-\infty, \infty)$. Assume that $y(x)$ is a solution of (5) and let

$$y(x) = y(e^t) = Y(t).$$

Using the chain-rule we deduce that

$$\frac{dy}{dx} = \frac{dY}{dt} \frac{dt}{dx} = \frac{1}{x} \frac{dY}{dt} \implies x \frac{dy}{dx} = \frac{dY}{dt}.$$  

Similarly, we deduce that

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{dY}{dt} \frac{dt}{dx} \right]$$

$$= \frac{d}{dx} \left[ \frac{dY}{dt} \right] \frac{dt}{dx} + \frac{dY}{dt} \frac{d}{dx} \left[ \frac{dt}{dx} \right]$$

$$= \left( \frac{d^2Y}{dt^2} \frac{dt}{dx} \right) \frac{dt}{dx} + \frac{dY}{dt} \frac{d^2t}{dx^2}$$

$$= \frac{1}{x^2} \frac{d^2Y}{dt^2} - \frac{1}{x^2} \frac{dY}{dt}.$$  

Therefore,

$$x^2 \frac{d^2y}{dx^2} = \frac{d^2Y}{dt^2} - \frac{dY}{dt}. $$

By substitution in (5) we deduce

$$a \left( Y''(t) - Y'(t) \right) + bY'(t) + cY(t) = 0$$

which simplifies to

$$aY''(t) + (b - a)Y'(t) + cY(t) = 0.$$  

(6)

We conclude that equation (5) is equivalent to the constant coefficients equation (6) in the sense that

$$Y(t) \text{ is a solution of (6)} \iff y(x) = Y(\ln x) \text{ is a solution of (5)}.$$