Partial Fractions

It is easy to combine two fractions into one such as

$$\frac{1}{x+3} + \frac{2}{x-4} = \frac{3x+2}{(x+3)(x-4)}.$$

The method of partial fractions is about the reverse process. We want to find constants A and B such that

$$\frac{3x+2}{(x+3)(x-4)} = \frac{A}{x+3} + \frac{B}{x-4}.$$

To decompose a rational function into partial fractions, we proceed as follows.

1. If the degree of the numerator $N(x)$ is greater than or equal to the degree of the denominator $D(x)$, use long division to obtain

$$\frac{N(x)}{D(x)} = P(x) + \frac{N_1(x)}{D(x)}$$

   where the degree of $N_1(x)$ is less than the degree of $D(x)$. Apply the next steps to $\frac{N_1(x)}{D(x)}$.

2. Completely factor the denominator $D(x)$ into factors of the form

$$(px+q)^m \quad \text{and} \quad (ax^2+bx+c)^n$$

   where ax^2+bx+c is an *irreducible quadratic.*

3. For *each* factor of the form $(px+q)^m$, the partial fractions decomposition must include the following m terms.

$$\frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}$$

4. For *each* factor of the form $(ax^2+bx+c)^n$, the partial fractions decomposition must include the following n terms.

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

5. Find the values of all the constants.

When the denominator has distinct roots, a useful trick to find the constants in a partial fractions decomposition is the cover-up method. Let's illustrate it with an example.

Consider the following partial fractions decomposition.

$$\frac{x^2+5x+4}{(x-1)(x-3)(x+2)} = \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{x+2} \qquad (*)$$

To find A, we could use the following two steps.

1. Multiply both sides of equation $(*)$ by $(x-1)(x-3)(x+2)$ to get

$$x^2+5x+4 = A(x-3)(x+2) + B(x-1)(x+1) + C(x-1)(x-3).$$

2. Set $x=1$ and solve for A.

$$1+5+4 = A(-2)(3)+0+0 \implies A = -\frac{5}{3}$$

The cover-up method combines these two steps in a single one as follows. To find A, we cover up $(x-1)$ and set $x=1$ in the left-hand side of $(*)$.

$$A = \left.\frac{x^2+5x+4}{\cancel{(x-1)}(x-3)(x+2)}\right|_{x=1} = -\frac{5}{3}$$

We can find constants B and C in the same way. To find B, we cover up $(x-3)$ and set $x=3$.

$$B = \left.\frac{x^2+5x+4}{(x-1)\cancel{(x-3)}(x+2)}\right|_{x=3} = \frac{14}{5}$$

To find C, we cover up $(x+2)$ and set $x=-2$.

$$C = \left.\frac{x^2+5x+4}{(x-1)(x-3)\cancel{(x+2)}}\right|_{x=-2} = -\frac{2}{15}$$