Evaluating $\int_{-\infty}^{\infty} e^{-x^2}\, dx$

Gilles Cazelais

The following funny story is found in [1].

> Once when lecturing in class, Lord Kelvin used the word *mathematician* and then interrupting himself asked his class: 'Do you know what a mathematician is?' Stepping to his blackboard he wrote upon it:
>
> $$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$
>
> Then putting his finger on what he had written, he turned to his class and said, 'a mathematician is one to whom that is as obvious as that twice two makes four is to you.'

We can evaluate the above integral as follows. Let

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx,$$

then

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy. \end{aligned}$$

Let's now use polar coordinates to evaluate the double integral. The region of integration

$$-\infty < x < \infty, \quad -\infty < y < \infty$$

becomes

$$0 \le r < \infty, \quad 0 \le \theta < 2\pi$$

in polar coordinates. Since $x^2 + y^2 = r^2$ and $dx\, dy = r\, dr\, d\theta$, we have

$$\begin{aligned} I^2 &= \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\, dr\, d\theta \\ &= 2\pi \int_0^{\infty} re^{-r^2}\, dr \\ &= 2\pi \left(-\frac{e^{-r^2}}{2}\right)\Bigg|_0^{\infty} \\ &= \pi\left(e^0 - \lim_{r\to\infty} e^{-r^2}\right) \\ &= \pi. \end{aligned}$$

By taking the square root, we obtain

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

References

[1] S.P. Thompson, *The Life of Lord Kelvin*, Chelsea Publishing, 2000.