Integration in Polar Coordinates

Consider the following double integral.

$$I = \int_0^2 \int_x^{\sqrt{8-x^2}} x^2 \sqrt{x^2+y^2}\; dy\, dx$$

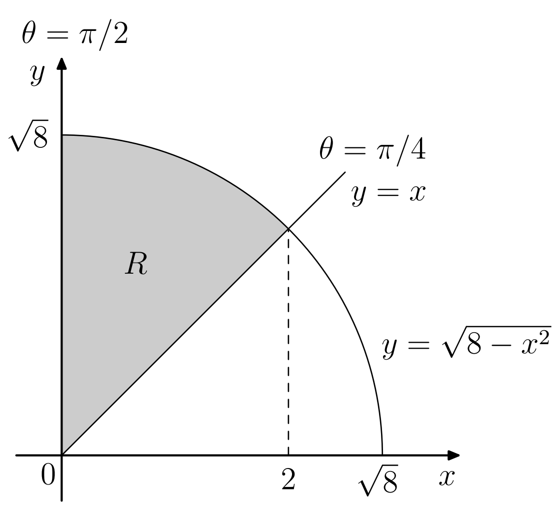
Evaluating it in rectangular coordinates is very tedious but if we switch to polar coordinates, it becomes much easier.

The double integral is

$$I = \iint_R x^2 \sqrt{x^2+y^2}\; dA$$

where the region of integration is

$$R = \{(x,y) \mid 0 \le x \le 2,\ x \le y \le \sqrt{8-x^2}\}.$$



If we switch to polar coordinates, the region R can be expressed as follows.

$$R = \left\{(r,\theta) \mid \tfrac{\pi}{4} \le \theta \le \tfrac{\pi}{2},\ 0 \le r \le \sqrt{8}\right\}$$

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In general, a rectangular function $f(x, y)$ becomes $f(r\cos\theta, r\sin\theta)$ in polar coordinates. In our case, we have

$$x^2\sqrt{x^2+y^2} = (r\cos\theta)^2 r = r^3\cos^2\theta.$$

Since in polar coordinates: $dA = r\,dr\,d\theta$, then

$$I = \int_{\pi/4}^{\pi/2}\int_0^{\sqrt{8}} (r^3\cos^2\theta)\, r\, dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/2}\int_0^{\sqrt{8}} r^4\cos^2\theta\, dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left(\frac{r^5}{5}\cos^2\theta\right)\Bigg|_{r=0}^{r=\sqrt{8}} d\theta$$

$$= \frac{(\sqrt{8})^5}{5}\int_{\pi/4}^{\pi/2} \cos^2\theta\, d\theta$$

$$= \frac{(\sqrt{8})^5}{10}\int_{\pi/4}^{\pi/2} (1+\cos 2\theta)\ d\theta$$

$$= \frac{(\sqrt{8})^5}{10}\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_{\pi/4}^{\pi/2}$$

$$= \frac{(\sqrt{8})^5}{10}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$\approx 5.166$$