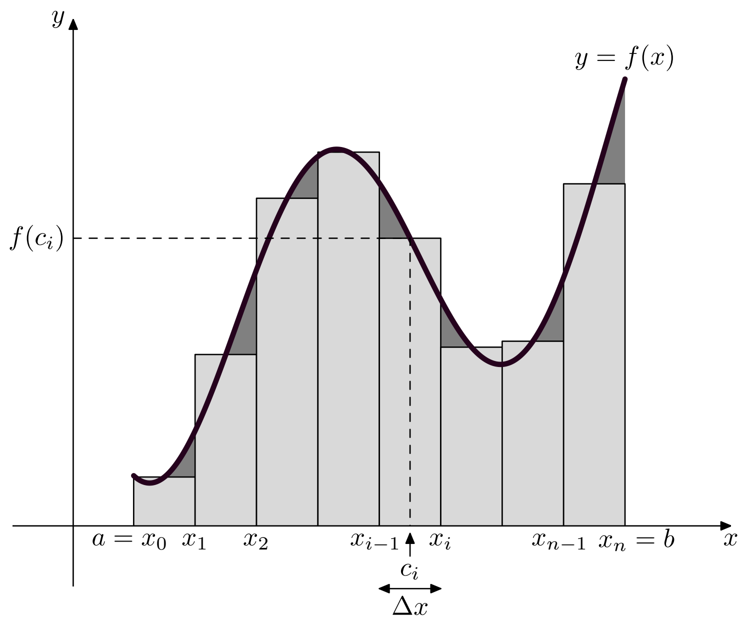
Fundamental Theorem of Calculus

Definition of a Definite Integral.
Let f be a continuous function on a closed interval $[a,b]$. We divide $[a,b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. Let $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$ be the endpoints of these subintervals. If c_i is any point in the subinterval $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$, then the definite integral of f from a to b is defined as follows.

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x}$$



Fundamental Theorem of Calculus. *Let f be a continuous function on the closed interval $[a,b]$. Then,*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, i.e., $F' = f$.

The following notation is often used.

$$F(x)\Big|_a^b = F(b) - F(a)$$

Example.
Since $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$, then

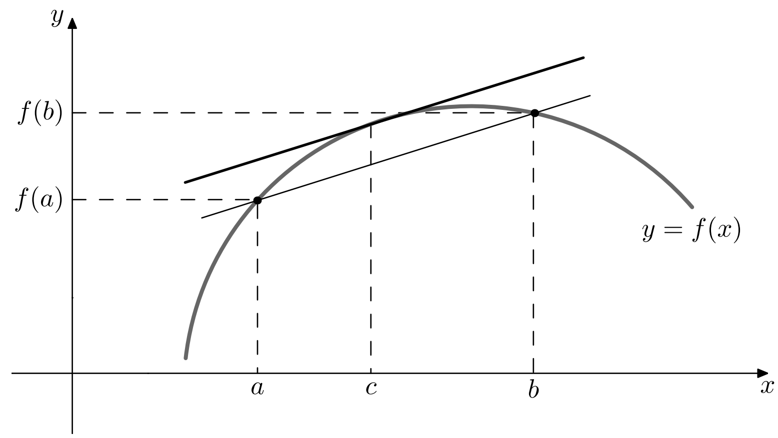
$$\int_1^2 x^2\,dx = \frac{1}{3}x^3\Big|_1^2 = \frac{1}{3}\,(2)^3 - \frac{1}{3}\,(1)^3 = \frac{7}{3}.$$

Gilles Cazelais. Typeset with LaTeX on January 3, 2007.

The proof of the Fundamental Theorem of Calculus requires the Mean Value Theorem.

Mean Value Theorem. *If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Let's now prove the Fundamental Theorem.

Proof. Let f be a continuous function on the interval $[a, b]$, and let F be an antiderivative of f. Let n be a positive integer and divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. Let $x_0 = a, x_1, x_2, \ldots, x_n = b$ be the endpoints of these subintervals. Then,

$$\begin{aligned}
F(b) - F(a) &= F(x_n) - F(x_0) \\
&= F(x_n) + \big(-F(x_{n-1}) + F(x_{n-1})\big) + \cdots + \big(-F(x_1) + F(x_1)\big) - F(x_0) \\
&= \big(F(x_n) - F(x_{n-1})\big) + \cdots + \big(F(x_1) - F(x_0)\big) \\
F(b) - F(a) &= \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big).
\end{aligned}$$

By the Mean Value Theorem, there exists a number c_i in each subinterval $[x_{i-1}, x_i]$ such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = \frac{F(x_i) - F(x_{i-1})}{\Delta x}.$$

Then,

$$F(x_i) - F(x_{i-1}) = F'(c_i)\Delta x.$$

Since F is an antiderivative of f, we have $F'(c_i) = f(c_i)$, therefore

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x.$$

By taking the limit as $n \to \infty$, we obtain

$$F(b) - F(a) = \int_a^b f(x)\, dx.$$

□