Definition of a Definite Integral.
Let $f$ be a continuous function on a closed interval $[a, b]$. We divide $[a, b]$ into $n$ subintervals of equal width $\Delta x = (b - a)/n$. Let $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$ be the endpoints of these subintervals. If $c_i$ is any point in the subinterval $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$, then the definite integral of $f$ from $a$ to $b$ is defined as follows.

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x$$

Fundamental Theorem of Calculus. Let $f$ be a continuous function on the closed interval $[a, b]$. Then,

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

where $F$ is any antiderivative of $f$, i.e., $F' = f$.

The following notation is often used.

$$F(x) \bigg|_a^b = F(b) - F(a)$$

Example.
Since $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$, then

$$\int_1^2 x^2 \, dx = \left[ \frac{1}{3}x^3 \right]_1^2 = \frac{1}{3}(2)^3 - \frac{1}{3}(1)^3 = \frac{7}{3}$$
The proof of the Fundamental Theorem of Calculus requires the Mean Value Theorem.

**Mean Value Theorem.** If \( f \) is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\), then there exists a number \( c \) in \((a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

Let’s now prove the Fundamental Theorem.

**Proof.** Let \( f \) be a continuous function on the interval \([a, b]\), and let \( F \) be an antiderivative of \( f \). Let \( n \) be a positive integer and divide \([a, b]\) into \( n \) subintervals of equal width \( \Delta x = (b - a)/n \). Let \( x_0 = a, x_1, x_2, \ldots, x_n = b \) be the endpoints of these subintervals. Then,

\[
F(b) - F(a) = F(x_n) - F(x_0)
= F(x_n) + (F(x_{n-1}) - F(x_{n-1})) + \cdots + (F(x_1) - F(x_1)) - F(x_0)
= \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})).
\]

By the Mean Value Theorem, there exists a number \( c_i \) in each subinterval \([x_{i-1}, x_i]\) such that

\[
F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = \frac{F(x_i) - F(x_{i-1})}{\Delta x}.
\]

Then,

\[
F(x_i) - F(x_{i-1}) = F'(c_i)\Delta x.
\]

Since \( F \) is an antiderivative of \( f \), we have \( F'(c_i) = f(c_i) \), therefore

\[
F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x.
\]

By taking the limit as \( n \to \infty \), we obtain

\[
F(b) - F(a) = \int_{a}^{b} f(x) \, dx.
\]

\( \square \)