Second partial derivatives test

Assume that \( f(x, y) \) is a function such that all its second partial derivatives are continuous on an open disk centered at \((x_0, y_0)\).

Also assume that \((x_0, y_0)\) is a critical point of \( f \), i.e.,

\[
\nabla f(x_0, y_0) = 0 \iff f_x(x_0, y_0) = 0 \text{ and } f_y(x_0, y_0) = 0.
\]

To classify a critical point we can use the following second derivative test.

Consider the Hessian matrix

\[
H(x, y) = \begin{bmatrix}
 f_{xx}(x, y) & f_{xy}(x, y) \\
 f_{yx}(x, y) & f_{yy}(x, y)
\end{bmatrix}
\]

and let \( D = \det(H(x_0, y_0)) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2 \).

1. If \( D > 0 \) and \( f_{xx}(x_0, y_0) > 0 \), then \((x_0, y_0)\) corresponds to a relative minimum of \( f \).
2. If \( D > 0 \) and \( f_{xx}(x_0, y_0) < 0 \), then \((x_0, y_0)\) corresponds to a relative maximum of \( f \).
3. If \( D < 0 \), then \((x_0, y_0)\) corresponds to a saddle point.
4. If \( D = 0 \), we can’t conclude anything.

Let’s use the test for \( f(x, y) = x^2 - y^2 \). Since

\[
f_x = 2x \quad \text{and} \quad f_y = -2y
\]

we see that \((0, 0)\) is the only critical point of \( f \). We also have that

\[
H(x, y) = \begin{bmatrix}
 2 & 0 \\
 0 & -2
\end{bmatrix}
\]

Since \( D = \det(H(0, 0)) = -4 < 0 \), we conclude that \((0, 0)\) corresponds to a saddle point.