Second partial derivatives test

Assume that $f(x, y)$ is a function such that all its second partial derivatives are continuous on an open disk centered at (x_0, y_0).

Also assume that (x_0, y_0) is a **critical point** of f, i.e.,

$$\nabla f(x_0, y_0) = \mathbf{0} \quad \Longleftrightarrow \quad f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0.$$

To classify a critical point we can use the following second derivative test.

Consider the Hessian matrix

$$H(x, y) = \begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix}$$

and let $D = \det(H(x_0, y_0)) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2$.

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) corresponds to a **relative minimum** of f.
2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) corresponds to a **relative maximum** of f.
3. If $D < 0$, then (x_0, y_0) corresponds to a **saddle point**.
4. If $D = 0$, we can't conclude anything.

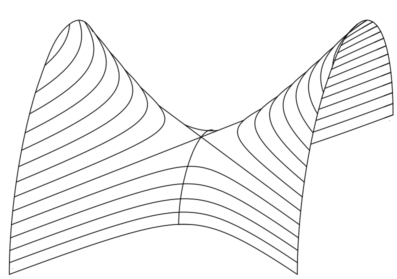
Let's use the test for $f(x, y) = x^2 - y^2$. Since

$$f_x = 2x \quad \text{and} \quad f_y = -2y$$

we see that $(0, 0)$ is the only critical point of f. We also have that

$$H(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

Since $D = \det(H(0, 0)) = -4 < 0$, we conclude that $(0, 0)$ corresponds to a saddle point.



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