Solutions of First Order ODEs

1. An equation that can be written in the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

is said to be **separable**. It can be solved by integration as follows.

$$\int g(y)\,dy = \int f(x)\,dx$$

2. The standard form of a **linear** first order ODE is

$$\frac{dy}{dx} + p(x)y = f(x).$$

It can be solved by multiplying it by the following integrating factor.

$$\mu(x) = e^{\int p(x)\,dx}$$

The solution of the equation is then

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)f(x)\,dx + C\right).$$

3. An equation expressed in differential form as

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be **exact** if the left-hand side of the equation corresponds to the differential of a function $f(x,y)$, i.e., if

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = M(x,y)\,dx + N(x,y)\,dy.$$

A necessary and sufficient condition for exactness is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

The solution method of an exact differential equation is the following.

(i) $f(x,y) = \int M(x,y)\,dx + g(y)$.

(ii) Find $g(y)$ using $f_y(x,y) = N(x,y)$.

(iii) The solution is $f(x,y) = c$.

If the equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact, we might be able to find an integrating factor to transform the equation in exact form.

- If $(M_y - N_x)/N = f(x)$, then an integrating factor of the equation is

$$\mu(x) = e^{\int f(x)\,dx}$$

- If $(N_x - M_y)/M = g(y)$, then an integrating factor of the equation is

$$\mu(y) = e^{\int g(y)\,dy}$$

4. An equation that can be written in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

is said to be **homogeneous**. It can be reduced to a separable equation by using the substitution $y = ux$. The equation then becomes

$$\frac{du}{dx} = \frac{F(u) - u}{x}.$$

5. An equation that can be expressed in the form

$$\frac{dy}{dx} + p(x)y = f(x)y^n$$

is called a **Bernoulli** equation. Note that it is linear if $n = 0$ or 1. For any other value of n, it can be reduced to the linear equation

$$\frac{du}{dx} + (1-n)p(x)u = (1-n)f(x)$$

by using the substitution

$$u = y^{1-n}.$$

6. An equation of the form

$$\frac{dy}{dx} = f(ax + by + c), \quad b \neq 0$$

can be reduced to the separable equation

$$\frac{du}{dx} = a + bf(u)$$

by using the substitution

$$u = ax + by + c.$$