Linear Combination of Sine and Cosine

Any linear combination of a cosine and a sine of equal periods is equal to a single sine with the same period but with a phase shift and a different amplitude.

In other words, given any c_1 and c_2, we can find A and ϕ such that

$$c_1 \cos \omega t + c_2 \sin \omega t = A \sin(\omega t + \phi). \tag{1}$$

We will now show how to find A and ϕ.

Using the identity

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

we deduce that (1) is equivalent to

$$c_1 \cos \omega t + c_2 \sin \omega t = A \cos \phi \sin \omega t + A \sin \phi \cos \omega t.$$

By setting equal the coefficients of $\cos \omega t$ and $\sin \omega t$, we obtain

$$A \sin \phi = c_1 \quad \text{and} \quad A \cos \phi = c_2.$$

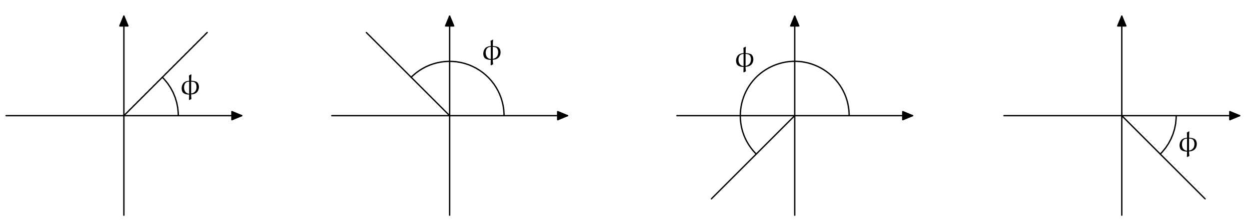
Observe that if $c_2 = 0$, then

$$A = c_1 \text{ and } \phi = \pi/2.$$

If $c_2 \neq 0$, we can find A and ϕ using

$$A = \sqrt{c_1^2 + c_2^2} \quad \text{and} \quad \tan \phi = \frac{c_1}{c_2}.$$

To find ϕ, we need to first identify its quadrant. The four cases are shown below.



$$\phi = \arctan\left(\frac{c_1}{c_2}\right) \qquad \phi = \pi + \arctan\left(\frac{c_1}{c_2}\right) \qquad \phi = \pi + \arctan\left(\frac{c_1}{c_2}\right) \qquad \phi = \arctan\left(\frac{c_1}{c_2}\right)$$

Example. Express $2\cos(3t) - 5\sin(3t)$ as $A \sin(\omega t + \phi)$.

Solution. First observe that $\omega = 3$. Since $c_1 = 2$ and $c_2 = -5$, then

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{2^2 + (-5)^2} = \sqrt{29}.$$

To find the quadrant of ϕ, observe that

$$\cos \phi = \frac{c_2}{A} < 0 \quad \text{and} \quad \sin \phi = \frac{c_1}{A} > 0.$$

We see that ϕ is a second-quadrant angle, therefore

$$\phi = \pi + \arctan\left(\frac{c_1}{c_2}\right) = \pi + \arctan\left(\frac{2}{-5}\right) \approx 2.761.$$

We conclude that

$$2\cos(3t) - 5\sin(3t) = \sqrt{29}\, \sin(3t + \arctan(-2/5) + \pi) \approx \sqrt{29}\, \sin(3t + 2.761).$$

Gilles Cazelais. Typeset with LaTeX on February 18, 2007.