Linear Combination of Sine and Cosine

Any linear combination of a cosine and a sine of equal periods is equal to a single sine with the same period but with a phase shift and a different amplitude.

In other words, given any $c_1$ and $c_2$, we can find $A$ and $\phi$ such that
\[ c_1 \cos \omega t + c_2 \sin \omega t = A \sin(\omega t + \phi). \] (1)

We will now show how to find $A$ and $\phi$.

Using the identity
\[ \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \]
we deduce that (1) is equivalent to
\[ c_1 \cos \omega t + c_2 \sin \omega t = A \cos \phi \sin \omega t + A \sin \phi \cos \omega t. \]

By setting equal the coefficients of $\cos \omega t$ and $\sin \omega t$, we obtain
\[ A \sin \phi = c_1 \quad \text{and} \quad A \cos \phi = c_2. \]

Observe that if $c_2 = 0$, then $A = c_1$ and $\phi = \pi/2$.

If $c_2 \neq 0$, we can find $A$ and $\phi$ using
\[ A = \sqrt{c_1^2 + c_2^2} \quad \text{and} \quad \tan \phi = \frac{c_1}{c_2}. \]

To find $\phi$, we need to first identify its quadrant. The four cases are shown below.

- $\phi = \arctan\left(\frac{c_1}{c_2}\right)$
- $\phi = \pi + \arctan\left(\frac{c_1}{c_2}\right)$
- $\phi = \pi + \arctan\left(\frac{c_1}{c_2}\right)$
- $\phi = \arctan\left(\frac{c_1}{c_2}\right)$

Example. Express $2 \cos(3t) - 5 \sin(3t)$ as $A \sin(\omega t + \phi)$.

Solution. First observe that $\omega = 3$. Since $c_1 = 2$ and $c_2 = -5$, then
\[ A = \sqrt{c_1^2 + c_2^2} = \sqrt{2^2 + (-5)^2} = \sqrt{29}. \]

To find the quadrant of $\phi$, observe that
\[ \cos \phi = \frac{c_2}{A} < 0 \quad \text{and} \quad \sin \phi = \frac{c_1}{A} > 0. \]

We see that $\phi$ is a second-quadrant angle, therefore
\[ \phi = \pi + \arctan\left(\frac{c_1}{c_2}\right) = \pi + \arctan\left(\frac{2}{-5}\right) \approx 2.761. \]

We conclude that
\[ 2 \cos(3t) - 5 \sin(3t) = \sqrt{29} \sin(3t + \arctan(-2/5) + \pi) \approx \sqrt{29} \sin(3t + 2.761). \]