Linear Algebra with MATLAB

Gilles Cazelais

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MATLAB makes it easy to perform computations with vectors and matrices. In this document, we introduce basic MATLAB commands for linear algebra and illustrate them with some examples.

1 Vectors

We can create a row vector.

```
>> v = [3 5 2]
v =
    3   5   2
```

A column vector is created in a similar way except that semicolons are used to separate the entries.

```
>> u = [2; 4; 1]
u =
    2
    4
    1
```

We can perform basic arithmetic of vectors. Note that ending a line with a semicolon suppresses printing of the output.

```
>> a = [1 2 3];
>> b = [1 3 5];

>> a+b
ans =
     2   5   8

>> b-a
ans =
     0   1   2

>> 3*a
ans =
     3   6   9
```

We can compute the norm and dot product of vectors, and we can compute the cross product of two vectors in $\mathbb{R}^3$.

```
>> norm(a)
ans =  3.7417

>> dot(a,b)
ans =  22
```

```
>> cross(a,b)
ans =
     1  -2  1
```

Example 1. Find the angle $0° \leq \theta \leq 180°$ between the vectors $\mathbf{u} = \begin{bmatrix} 3, & 2, & -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1, & -1, & 4 \end{bmatrix}$.

Solution: We use the formula

$$\theta = \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right).$$

The MATLAB command `acos` returns an angle in radians and the command `acosd` returns an angle in degrees.

```
>> u = [3 2 -1];
>> v = [1 -1 4];

>> theta = acosd( dot(u,v) / (norm(u)*norm(v)) )
theta =  100.89
```

The answer is $\theta = 100.89°$.

Example 2. Find a vector perpendicular to the plane passing through the three points

$$A = (0, 1, 2), \quad B = (2, 3, 1), \quad \text{and} \quad C = (4, 5, 2).$$

Solution: Such a vector is $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$.

```
>> a = [0 1 2];
>> b = [2 3 1];
>> c = [1 5 2];

>> ab = b-a
ab =
     2  2  -1

>> ac = c-a
ac =
     1  4  0

>> n = cross(ab, ac)
n =
     4  -1  6
```

Example 3. Find the projection of $\mathbf{v} = \begin{bmatrix} 1, & 2, & 3 \end{bmatrix}$ onto $\mathbf{u} = \begin{bmatrix} 2, & 3, & 1 \end{bmatrix}$.

Solution: The projection is obtained by using

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u}.$$

```
>> v = [1 2 3];
>> u = [2 3 1];

>> proj = (dot(u,v)/dot(u,u))*u
proj =
       1.57143   2.35714   0.78571
```

If rational answers are preferred, we use `format rat`.

```
>> format rat
>> proj = (dot(u,v)/dot(u,u))*u
proj =
       11/7    33/14    11/14
```

2 Matrices

We can create a matrix by using semicolons to separate the rows.

```
>> A = [1 2; 3 4; 5 6]
A =
    1   2
    3   4
    5   6
```

We can determine the size of a matrix.

```
>> size(A)
ans =
     3   2
```

We can find the transpose of a matrix.

```
>> A'
ans =
     1   3   5
     2   4   6
```

We can perform basic arithmetic of matrices.

```
>> A = [1 2; 3 4];
>> B = [2 1; 5 3];

>> 5*A+2*B
ans =
       9   12
      25   26

>> A*B
ans =
     12    7
     26   15

>> A^3
ans =
      37    54
      81   118
```

We can compute the determinant and find the inverse of a square matrix.

```
>> A = [1 2 -1; 2 2 4; 1 3 -3];

>> det(A)
ans =
      -2

>> format rat
>> inv(A)
ans =
         9    -3/2    -5
        -5       1     3
        -2     1/2     1
```

We can obtain the reduced row echelon form of a matrix.

```
>> A = [1 2 8; 3 1 9]
A =
    1   2   8
    3   1   9

>> rref(A)
ans =
      1    0    2
      0    1    3
```

We can create an $n \times n$ identity matrix with the command `eye(n)`.

```
>> I = eye(3)
I =
    1   0   0
    0   1   0
    0   0   1
```

The command `diag` can be used to quickly create a diagonal matrix.

```
>> D = diag([4 2 7])
D =
    4   0   0
    0   2   0
    0   0   7
```

An $m \times n$ zero matrix can be created with the command `zeros(m,n)`.

```
>> zeros(2,3)
ans =
      0   0   0
      0   0   0
```

We can get the "ij" entry of a matrix A by using the command `A(i,j)`.

```
>> A = [1 2 3; 4 5 6; 7 8 9]
A =
    1   2   3
    4   5   6
    7   8   9

>> A(2,3)
ans = 6
```

We can extract the n^{th} row of A with `A(n,:)` and the m^{th} column of A with `A(:,m)`.

```
>> A(2,:)
ans =
      4   5   6

>> A(:,3)
ans =
      3
      6
      9
```

Example 4. Find the solution of the following linear system.

$$\begin{cases} x + y + z = 2 \\ -x + z = 1 \\ 2x + 3y + 5z = 9 \end{cases}$$

Solution: First, we define the augmented matrix of the system.

```
>> A = [1 1 1 2; -1 0 1 1; 2 3 5 9]
A =
    1   1   1   2
   -1   0   1   1
    2   3   5   9
```

Next, we find the reduced row echelon form of the augmented matrix.

```
>> rref(A)
ans =
      1   0   0   1
      0   1   0  -1
      0   0   1   2
```

We see that the solution of the system is $x = 1$, $y = -1$, and $z = 2$.

Example 5. Find the vector obtained if we rotate around the origin the vector $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ by an angle of $\pi/3$ counterclockwise.

Solution: The matrix corresponding to a counterclockwise rotation of angle θ around the origin is given by

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The vector obtained after rotation is

$$\mathbf{v} = R(\pi/3)\mathbf{u}.$$

```
>> theta = pi/3;
>> R = [cos(theta)  -sin(theta); sin(theta)  cos(theta)]
R =
    0.50000  -0.86603
    0.86603   0.50000

>> u = [2; 3];
>> v = R*u
v =
    -1.5981
     3.2321
```

Example 6. Find the least squares solution of the following linear system.

$$\begin{cases} 3x + y = 4 \\ x + y = 1 \\ x + 2y = 3 \end{cases}$$

Solution: For a linear system in matrix form $A\mathbf{x} = \mathbf{b}$ (where A has linearly independent columns), the least squares solution is given by

$$\overline{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}.$$

```
>> A = [3 1; 1 1; 1 2]
A =
    3  1
    1  1
    1  2

>> b = [4; 1; 3]
b =
    4
    1
    3

>> inv(A'*A)*A'*b
ans =
      1.00000
      0.83333
```

MATLAB offers the shortcut `A\b` to obtain the least squares solution of a system $A\mathbf{x} = \mathbf{b}$.

```
>> A\b
ans =
      1.00000
      0.83333

>> format rat
>> A\b
ans =
          1
        5/6
```

The least squares solution of the system is $x = 1$, $y = 5/6 \approx 0.83333$.

3 Eigenvalues and Eigenvectors

We can find the eigenvalues of a square matrix A with the command `eig(A)`.

```
>> A = [1 1 2; 1 2 1; 2 1 1]
A =
    1  1  2
    1  2  1
    2  1  1

>> eig(A)
ans =
      -1
       1
       4
```

The characteristic polynomial of matrix A is obtained with the command `poly(A)`.

```
>> poly(A)
ans =
      1  -4  -1  4
```

The characteristic polynomial is then

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0.$$

Although it is better to use the built-in command `eig`, an alternative method to find the eigenvalues of A is by finding the roots of the characteristic polynomial.

```
>> roots(poly(A))
ans =
      -1
       1
       4
```

In order to obtain the eigenvectors of A, we need to set two variables equal to `eig(A)`.

```
>> [P, D] = eig(A)
P =
         0.70711    0.40825    0.57735
     2.0777e-16    -0.8165    0.57735
        -0.70711    0.40825    0.57735

D =
    -1   0   0
     0   1   0
     0   0   4
```

The eigenvalues are on the diagonal of D and the corresponding eigenvectors are the columns of P. Note that MATLAB always returns the eigenvectors as unit vectors. For our example, the eigenvalues and eigenvectors of A are the following.

$$\lambda_1 = -1, \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \approx \begin{bmatrix} 0.70711 \\ 0 \\ -0.70711 \end{bmatrix}$$

$$\lambda_2 = 1, \quad \mathbf{v}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.40825 \\ -0.8165 \\ 0.40825 \end{bmatrix}$$

$$\lambda_3 = 4, \quad \mathbf{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.57735 \\ 0.57735 \\ 0.57735 \end{bmatrix}$$

We can verify that

$$A = PDP^{-1}.$$

```
>> P*D*inv(P)
ans =
      1   1   2
      1   2   1
      2   1   1
```

4 Complex Numbers

Complex numbers can be entered in MATLAB as follows.

```
>> z = 3 + 4i
z =  3 + 4i
```

We can find the real and imaginary part of a complex number.

```
>> real(z)
ans =  3

>> imag(z)
ans =  4
```

We can find the polar form of a complex number z with $r \geq 0$ and $-\pi < \theta \leq \pi$.

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

```
>> z = 3 + 4i;

>> r = abs(z)
r =  5

>> theta = arg(z)
theta =  0.92730

>> r*exp(i*theta)
ans =  3 + 4i
```

We can perform basic arithmetic of complex numbers.

```
>> z1 = 2 + 5i;
>> z2 = 3 - 2i;

>> z1+z2
ans =  5 + 3i

>> z1*z2
ans =  16 + 11i

>> z1/z2
ans = -0.30769 + 1.46154i

>> format rat
>> z1/z2
ans = -4/13 + 19/13i
```

We can use MATLAB to find numerical roots (real or complex) of any polynomial. For example, let's find the roots of

$$x^3 - 10x^2 + 41x - 50 = 0.$$

```
>> p = [1  -10  41  -50];
>> roots(p)
ans =
      4.0000 + 3.0000i
      4.0000 - 3.0000i
      2.0000 + 0.0000i
```

The roots are

$$4 + 3i, \quad 4 - 3i, \quad \text{and} \quad 2.$$

As we'll see in the following example, MATLAB can find complex eigenvalues and eigenvectors of a square matrix.

Example 7. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -2 \\ 4 & 12 & -5 \end{bmatrix}.$$

Solution: Let's first enter the matrix.

```
>> A = [1 2 -2; 2 5 -2; 4 12 -5]
A =
    1   2  -2
    2   5  -2
    4  12  -5
```

We can first find the eigenvalues.

```
>> eig(A)
ans =
       1 + 2i
       1 - 2i
      -1 + 0i
```

Let's now find the eigenvectors.

```
>> [P, D] = eig(A)
P =
    0.00000 + 0.40825i   0.00000 - 0.40825i   0.70711 + 0.00000i
    0.40825 - 0.00000i   0.40825 + 0.00000i   0.00000 + 0.00000i
    0.81650 + 0.00000i   0.81650 - 0.00000i   0.70711 + 0.00000i


D =
     1 + 2i        0        0
          0   1 - 2i        0
          0        0  -1 + 0i
```

We see that the eigenvalues and corresponding eigenvectors are

$$\lambda_1 = 1 + 2i, \quad \mathbf{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} i \\ 1 \\ 2 \end{bmatrix} \approx \begin{bmatrix} 0.40825i \\ 0.40825 \\ 0.81650 \end{bmatrix}$$

$$\lambda_2 = 1 - 2i, \quad \mathbf{v}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -i \\ 1 \\ 2 \end{bmatrix} \approx \begin{bmatrix} -0.40825i \\ 0.40825 \\ 0.81650 \end{bmatrix}$$

$$\lambda_3 = -1, \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.70711 \\ 0 \\ 0.70711 \end{bmatrix}$$

Example 8. Find the three cube roots of -27.

Solution: We have to find the roots of

$$x^3 + 27 = 0.$$

This is equivalent to

$$x^3 + 0x^2 + 0x + 27 = 0.$$

```
>> p = [1 0 0 27];
>> roots(p)
ans =
      -3.0000 + 0.0000i
       1.5000 + 2.5981i
       1.5000 - 2.5981i
```

The three roots are

$$-3, \quad \frac{3}{2} + \frac{3\sqrt{3}}{2}i \approx 1.5000 + 2.5981i, \quad \text{and} \quad \frac{3}{2} - \frac{3\sqrt{3}}{2}i \approx 1.5000 - 2.5981i.$$

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