

Gauss-Jordan Elimination Method

The following **row operations** on the augmented matrix of a system produce the augmented matrix of an equivalent system, i.e., a system with the same solution as the original one.

- Interchange any two rows.
- Multiply each element of a row by a nonzero constant.
- Replace a row by the sum of itself and a constant multiple of another row of the matrix.

For these row operations, we will use the following notations.

- $R_i \leftrightarrow R_j$ means: Interchange row i and row j .
- αR_i means: Replace row i with α times row i .
- $R_i + \alpha R_j$ means: Replace row i with the sum of row i and α times row j .

The Gauss-Jordan elimination method to solve a system of linear equations is described in the following steps.

1. Write the augmented matrix of the system.
2. Use row operations to transform the augmented matrix in the form described below, which is called the **reduced row echelon form** (RREF).
 - (a) The rows (if any) consisting entirely of zeros are grouped together at the bottom of the matrix.
 - (b) In each row that does not consist entirely of zeros, the leftmost nonzero element is a 1 (called a leading 1 or a pivot).
 - (c) Each column that contains a leading 1 has zeros in all other entries.
 - (d) The leading 1 in any row is to the left of any leading 1's in the rows below it.
3. Stop process in step 2 if you obtain a row whose elements are all zeros except the last one on the right. In that case, the system is inconsistent and has no solutions. Otherwise, finish step 2 and read the solutions of the system from the final matrix.

Note: When doing step 2, row operations can be performed in any order. Try to choose row operations so that as few fractions as possible are carried through the computation. This makes calculation easier when working by hand.

Example 1. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x + y + z = 5 \\ 2x + 3y + 5z = 8 \\ 4x + 5z = 2 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array} \right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array} \right] &\xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array} \right] \\ &\xrightarrow{R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array} \right] \\ &\xrightarrow{R_3+4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array} \right] \\ &\xrightarrow{\frac{1}{13}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array} \right] \\ &\xrightarrow{R_2-3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array} \right] \\ &\xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array} \right] \\ &\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array} \right] \end{aligned}$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{x = 3, \quad y = 4, \quad z = -2.}$$

Example 2. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x + 2y - 3z = 2 \\ 6x + 3y - 9z = 6 \\ 7x + 14y - 21z = 13 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array} \right]$$

Let's now perform row operations on this augmented matrix.

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array} \right] &\xrightarrow{R_2-6R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 7 & 14 & -21 & 13 \end{array} \right] \\ &\xrightarrow{R_3-7R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 0 & 0 & 0 & -1 \end{array} \right] \end{aligned}$$

We obtain a row whose elements are all zeros except the last one on the right. Therefore, we conclude that the system of equations is inconsistent, i.e., it has no solutions.

Example 3. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} 4y + z = 2 \\ 2x + 6y - 2z = 3 \\ 4x + 8y - 5z = 4 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array} \right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\begin{aligned} \left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array} \right] & \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 4 & 8 & -5 & 4 \end{array} \right] \\ & \xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & -4 & -1 & -2 \end{array} \right] \\ & \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \\ & \xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array} \right] \\ & \xrightarrow{R_1 - 6R_2} \left[\begin{array}{ccc|c} 2 & 0 & -7/2 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array} \right] \\ & \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & -7/4 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

This last matrix is in reduced row echelon form so we can stop. It corresponds to the augmented matrix of the following system.

$$\begin{cases} x - \frac{7}{4}z = 0 \\ y + \frac{1}{4}z = \frac{1}{2} \end{cases}$$

We can express the solutions of this system as

$$x = \frac{7}{4}z, \quad y = \frac{1}{2} - \frac{1}{4}z.$$

Since there is no specific value for z , it can be chosen arbitrarily. This means that there are **infinitely many** solutions for this system. We can represent all the solutions by using a parameter t as follows.

$$\boxed{x = \frac{7}{4}t, \quad y = \frac{1}{2} - \frac{1}{4}t, \quad z = t}$$

Any value of the parameter t gives us a solution of the system. For example,

$$t = 4 \quad \text{gives the solution} \quad (x, y, z) = (7, -\frac{1}{2}, 4)$$

$$t = -2 \quad \text{gives the solution} \quad (x, y, z) = (-\frac{7}{2}, 1, -2).$$

Example 4. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} A + B + 2C = 1 \\ 2A - B + D = -2 \\ A - B - C - 2D = 4 \\ 2A - B + 2C - D = 0 \end{cases}$$

Solution: We will perform row operations on the augmented matrix of the system until we obtain a matrix in reduced row echelon form.

$$\begin{aligned} & \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 2 & -1 & 0 & 1 & -2 \\ 1 & -1 & -1 & -2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{array} \right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 1 & -1 & -1 & -2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{array} \right] \xrightarrow{R_3-R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 2 & -1 & 2 & -1 & 0 \end{array} \right] \\ & \xrightarrow{R_4-2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -2 & -1 & -2 \end{array} \right] \xrightarrow{R_4-R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & 0 & 2 & -2 & 2 \end{array} \right] \\ & \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & 0 & 2 & -2 & 2 \end{array} \right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & 0 & 2 & -2 & 2 \end{array} \right] \\ & \xrightarrow{R_3+3R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1/2 & 4 & -17/2 \\ 0 & 0 & 2 & -2 & 2 \end{array} \right] \xrightarrow{2R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 2 & -2 & 2 \end{array} \right] \\ & \xrightarrow{R_4-2R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & -18 & 36 \end{array} \right] \xrightarrow{-\frac{1}{18}R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & 1 & -2 \end{array} \right] \\ & \xrightarrow{R_3-8R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array} \right] \xrightarrow{R_2-R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array} \right] \\ & \xrightarrow{R_2-\frac{3}{2}R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array} \right] \xrightarrow{R_1-2R_3, R_1-R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array} \right] \end{aligned}$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{A = 1, \quad B = 2, \quad C = -1, \quad D = -2.}$$