Review of Eigenvalues and Eigenvectors

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An **eigenvector** of a square matrix $\mathbf{A}$ is a nonzero vector $\mathbf{K}$ such that

$$\mathbf{AK} = \lambda\mathbf{K}$$

for some number λ called an **eigenvalue** of $\mathbf{A}$. The vector $\mathbf{K}$ is called an eigenvector corresponding to the eigenvalue λ.

To find the eigenvalues of $\mathbf{A}$ we find the roots of

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

To find the eigenvectors corresponding to eigenvalue λ, we solve the linear system

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = \mathbf{0}.$$

Example 1. Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 1 & -7 & 3 \\ -1 & -1 & 1 \\ 4 & -4 & 0 \end{bmatrix}.$$

Solution: We first find the eigenvalues of $\mathbf{A}$.

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 1-\lambda & -7 & 3 \\ -1 & -1-\lambda & 1 \\ 4 & -4 & -\lambda \end{vmatrix} \\ &= -\lambda^3 + 16\lambda \\ &= -\lambda(\lambda - 4)(\lambda + 4) \end{aligned}$$

The eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = 4, \quad \lambda_3 = -4.$$

Let's now find the eigenvectors.

For $\lambda_1 = 0$.

$$\mathbf{A} - \lambda_1\mathbf{I} = \begin{bmatrix} 1 & -7 & 3 \\ -1 & -1 & 1 \\ 4 & -4 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

If $\mathbf{K} = \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$, we see that the nonzero solutions of

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{K} = \mathbf{0}$$

satisfy

$$x = \tfrac{1}{2}t, \quad y = \tfrac{1}{2}t, \quad z = t, \quad \text{for any } t \neq 0.$$

By choosing $t = 2$, we obtain the eigenvector

$$\mathbf{K}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

For $\lambda_2 = 4$.

$$\mathbf{A} - \lambda_2\mathbf{I} = \begin{bmatrix} -3 & -7 & 3 \\ -1 & -5 & 1 \\ 4 & -4 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The nonzero solutions of $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{K} = \mathbf{0}$ satisfy

$$x = t, \quad y = 0, \quad z = t, \quad \text{for any } t \neq 0.$$

By choosing $t = 1$, we obtain the eigenvector

$$\mathbf{K}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda_3 = -4$.

$$\mathbf{A} - \lambda_3\mathbf{I} = \begin{bmatrix} 5 & -7 & 3 \\ -1 & 3 & 1 \\ 4 & -4 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The nonzero solutions of $(\mathbf{A} - \lambda_3\mathbf{I})\mathbf{K} = \mathbf{0}$ satisfy

$$x = -2t, \quad y = -t, \quad z = t, \quad \text{for any } t \neq 0.$$

By choosing $t = -1$, we obtain the eigenvector

$$\mathbf{K}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

Example 2. Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: We first find the eigenvalues of $\mathbf{A}$.

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -(\lambda-2)(\lambda+1)^2$$

The eigenvalues are

$$\lambda_1 = 2, \quad \lambda_2 = -1 \text{ (multiplicity 2)}.$$

Let's now find the eigenvectors.

For $\lambda_1 = 2$.

$$\mathbf{A}-\lambda_1\mathbf{I} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

If $\mathbf{K} = \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$, we see that the nonzero solutions of

$$(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{K} = \mathbf{0}$$

satisfy

$$x = t, \quad y = t, \quad z = t, \quad \text{for any } t \neq 0.$$

By choosing $t = 1$, we obtain the eigenvector

$$\mathbf{K}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = -1$.

$$\mathbf{A}-\lambda_1\mathbf{I} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The nonzero solutions of $(\mathbf{A}-\lambda_2\mathbf{I})\mathbf{K} = \mathbf{0}$ satisfy

$$x = -s-t, \quad y = s, \quad z = t, \quad \text{for any } (s,t) \neq (0,0).$$

We conclude that

$$\mathbf{K} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

are eigenvectors for any $(s,t) \neq (0,0)$. We have two linearly independent eigenvectors corresponding to $\lambda_2 = -1$ namely

$$\mathbf{K}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Example 3. Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -2 \\ 4 & 12 & -5 \end{bmatrix}.$$

Solution: We first find the eigenvalues of $\mathbf{A}$.

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 & -2 \\ 2 & 5-\lambda & -2 \\ 4 & 12 & -5-\lambda \end{vmatrix} = -\lambda^3+\lambda^2-3\lambda-5$$

Using a computer or a calculator to solve this cubic, we find that the eigenvalues are

$$\lambda_1 = -1, \quad \lambda_2 = 1+2i, \quad \lambda_3 = 1-2i.$$

Let's now find the eigenvectors.

For $\lambda_1 = -1$.

$$\mathbf{A}-\lambda_1\mathbf{I} = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 6 & -2 \\ 4 & 12 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If $\mathbf{K} = \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$, we see that the nonzero solutions of

$$(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{K} = \mathbf{0}$$

satisfy

$$x = t, \quad y = 0, \quad z = t, \quad \text{for any } t \neq 0.$$

By choosing $t = 1$, we obtain the eigenvector

$$\mathbf{K}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 1+2i$.

$$\mathbf{A}-\lambda_2\mathbf{I} = \begin{bmatrix} -2i & 2 & -2 \\ 2 & 4-2i & -2 \\ 4 & 12 & -6-2i \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -i/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

The nonzero solutions of $(\mathbf{A}-\lambda_2\mathbf{I})\mathbf{K} = \mathbf{0}$ satisfy

$$x = \tfrac{i}{2}t, \quad y = \tfrac{1}{2}t, \quad z = t, \quad \text{for any } t \neq 0.$$

By choosing $t = 2$, we obtain the eigenvector

$$\mathbf{K}_2 = \begin{bmatrix} i \\ 1 \\ 2 \end{bmatrix}.$$

For $\lambda_3 = 1-2i$, by taking the complex conjugate of $\mathbf{K_2}$ we obtain an eigenvector

$$\mathbf{K}_3 = \begin{bmatrix} -i \\ 1 \\ 2 \end{bmatrix}.$$

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