Balancing Chemical Equations

Consider the the following unbalanced chemical equation.

$$\mathrm{C_2H_6 + O_2 \to CO_2 + H_2O}$$

To balance the equation, we need to find four positive integers x_1, x_2, x_3, and x_4 so that the equation

$$x_1\,\mathrm{C_2H_6} + x_2\,\mathrm{O_2} \to x_3\,\mathrm{CO_2} + x_4\,\mathrm{H_2O}$$

is such that we have the same number of carbon (C), hydrogen (H), and oxygen (O) on each side of the equation. To satisfy this constraint, we deduce the following three linear equations.

$$\begin{aligned} \text{carbon} &\implies 2x_1 = x_3 \\ \text{hydrogen} &\implies 6x_1 = 2x_4 \\ \text{oxygen} &\implies 2x_2 = 2x_3 + x_4 \end{aligned}$$

We can solve the system of linear equation

$$\begin{aligned} 2x_1 - x_3 &= 0 \\ 6x_1 - 2x_4 &= 0 \\ 2x_2 - 2x_3 - x_4 &= 0 \end{aligned}$$

by Gauss-Jordan elimination.

$$\begin{bmatrix} 2 & 0 & -1 & 0 \\ 6 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & -7/6 \\ 0 & 0 & 1 & -2/3 \end{bmatrix}$$

The general solution of the linear system is

$$x_1 = \frac{1}{3}\,t, \quad x_2 = \frac{7}{6}\,t, \quad x_3 = \frac{2}{3}\,t, \quad x_4 = t, \quad \text{for all } t \in \mathbb{R}.$$

The smallest positive integer solution is obtained by choosing $t = 6$. In this case we obtain

$$x_1 = 2, \quad x_2 = 7, \quad x_3 = 4, \quad x_4 = 6.$$

We have obtained a balanced chemical equation.

$$\boxed{2\,\mathrm{C_2H_6} + 7\,\mathrm{O_2} \to 4\,\mathrm{CO_2} + 6\,\mathrm{H_2O}}$$