Math 251
Practice Questions

1. Consider the following two vectors in $\mathbb{R}^2$.

$$\mathbf{u} = \begin{bmatrix} x \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Find the values of x if

(a) $\mathbf{u}$ is parallel to $\mathbf{v}$, (b) $\mathbf{u}$ is perpendicular to $\mathbf{v}$, (c) the angle between $\mathbf{u}$ and $\mathbf{v}$ is $30°$.

Answers: (a) $x = 8/3$ (b) $x = -3/2$ (c) $x_1 \approx 16.6$ and $x_2 \approx 0.854$

2. (a) Find parametric equations for the line passing through the points

$$A = (2, 4, -1), \quad \text{and} \quad B = (5, 0, 7).$$

(b) At which point does the line intersect the xy-plane?

(c) Find the distance between the line in (a) and point $P = (1, 0, 1)$.

Answers: (a) $x = 2 + 3t,\ y = 4 - 4t,\ z = -1 + +8t$ (b) $(19/8, 7/2, 0)$ (c) ≈ 3.398

3. Consider the four points

$$A = (0, 1, 0), \quad B = (3, 2, 0), \quad C = (0, 3, -3), \quad \text{and} \quad P = (1, 1, 2).$$

(a) Find an equation in general form for the plane passing through A, B, and C.

(b) Find the distance between the plane in (a) and point P.

(c) Find the point Q on the plane in (a) that is closest to P.

(d) Find the area of the triangle with vertices at points A, B, and C.

Answers: (a) $-x + 3y + 2z = 3$ (b) $3\sqrt{14}/14 \approx 0.802$ (c) $Q = (17/14, 5/14, 11/7)$ (d) $\sqrt{126}/2$

4. Find a unit vector parallel to the xz-plane and perpendicular to vector

$$\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

Answers: $\pm\frac{1}{\sqrt{20}} \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}$

5. Find all solutions of the following linear system. Identify all leading and free variables.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 2x_4 &= 2 \\ 2x_1 + 5x_2 - 8x_3 + 6x_4 &= 5 \\ 3x_1 + 4x_2 - 5x_3 + 2x_4 &= 4 \end{aligned}$$

Answer: The leading variables are x_1 and x_2. The free variables are x_3 and x_4.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

6. Determine whether the lines

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$$

intersect and if they do, find their point of intersection.

Answer: The two lines intersect at point $(13, 8, 6)$, corresponding to $s = 4$ and $t = 2$.

7. Find the equation of the parabola passing through the three points:

$$(-1, 10), \ (1, 0), \ (2, 4).$$

Write your answer in the form $y = ax^2 + bx + c$.

Answer: $y = 3x^2 - 5x + 2$.

8. Find parametric equations for the line of intersection of the planes

$$3x + 2y - 4z = 6 \quad \text{and} \quad x - 3y - 2z = 4.$$

Answer: $x = \frac{26}{11} + \frac{16}{11}t, \quad y = -\frac{6}{11} - \frac{2}{11}t, \quad z = t$

9. Consider the following matrices.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 6 & 2 \\ 4 & 3 \end{bmatrix}$$

Evaluate: (a) $AB + 4C$ (b) BA (c) $A + B^T$ (d) C^{-1}

Answers: (a) $AB + 4C = \begin{bmatrix} 47 & 16 \\ 13 & 5 \end{bmatrix}$ (b) $BA = \begin{bmatrix} 7 & 5 & 5 \\ 10 & 4 & 8 \\ 10 & 15 & 5 \end{bmatrix}$

(c) $A + B^T = \begin{bmatrix} 5 & 7 & 6 \\ 2 & -2 & 2 \end{bmatrix}$ (d) $C^{-1} = \begin{bmatrix} 3/10 & -1/5 \\ -2/5 & 3/5 \end{bmatrix}$

10. Consider the following matrix.

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ -3 & 3 & 4 \end{bmatrix}$$

(a) Find its inverse A^{-1}.

(b) Use your answer in (a) to solve the following system of linear equations.

$$\begin{cases} x - y - z = 2 \\ -y + z = 1 \\ -3x + 3y + 4z = 3 \end{cases}$$

Answers: (a) $A^{-1} = \begin{bmatrix} 7 & -1 & 2 \\ 3 & -1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$ (b) $x = 19,\ y = 8,\ z = 9$

11. Solve for the matrix X in the following matrix equation.

$$(A^{-1}X)^{-1} = (AB^{-1})^{-1}(AB^2).$$

Simplify your answer as much as possible.

Answer: $X = A(B^{-1})^3$

12. Find the matrix A that satisfies the following.

$$\left(2A^{-1} + I_2\right)^T = \begin{bmatrix} 9 & 4 \\ 2 & 3 \end{bmatrix}$$

Answer: $A = \begin{bmatrix} 1/2 & -1/2 \\ -1 & 2 \end{bmatrix}$

13. Consider the matrix

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$$

(a) Write A^{-1} as a product of elementary matrices.

(b) Write A as a product of elementary matrices.

Answers: Lots of answers are possible.

(a) $A^{-1} = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

14. Consider the following vectors.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ linearly dependent or linearly independent?

Answer: They are linearly dependent (LD). Observe that $\mathbf{v}_3 = 3\mathbf{v}_1 - \mathbf{v}_2$.

15. Determine if the three vectors in $\mathbb{R}^4$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 5 \\ 6 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

are linearly dependent or linearly independent.

Answer: They are linearly independent (LI).

16. Consider the following three vectors in $\mathbb{R}^3$.

$$\mathbf{u} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Is vector $\mathbf{u}$ in $\text{span}(\mathbf{v_1}, \mathbf{v_2})$? If yes, express $\mathbf{u}$ as a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$.

Answer: The vector $\mathbf{u}$ is not in $\text{span}(\mathbf{v_1}, \mathbf{v_2})$.

17. Find an LU factorization of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 2 & 3 \\ -4 & -2 & 1 \end{bmatrix}$$

Answer: Lots of answers are possible.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 9 \\ 0 & 0 & 16 \end{bmatrix}$$

18. Consider the following 3×5 matrix.

$$A = \begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$$

(a) Find bases for the subspaces $\text{row}(A)$, $\text{col}(A)$, and $\text{null}(A)$.

(b) What are the values of $\text{rank}(A)$ and $\text{nullity}(A)$?

(c) What are all the possible values of $\text{rank}(A)$ and $\text{nullity}(A)$ for an arbitrary 3×5 matrix?

Answers:

(a) Basis of $\text{row}(A)$:

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 1/2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 3 & 7/2 \end{bmatrix}$$

Basis of $\text{col}(A)$:

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Basis of null(A):

$$\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}, \quad \begin{bmatrix}-1\\0\\-3\\1\\0\end{bmatrix}, \quad \begin{bmatrix}-1/2\\0\\-7/2\\0\\1\end{bmatrix}.$$

(b) rank(A) $= 2$ and nullity(A) $= 3$

(c) The possible values are the following.

rank(A)	null(A)
0	5
1	4
2	3
3	2

19. Consider the following four vectors in $\mathbb{R}^3$.

$$\mathbf{u} = \begin{bmatrix}1\\2\\5\end{bmatrix}, \ \mathbf{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

(a) Show that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form a basis of $\mathbb{R}^3$.

(b) Find the coordinate vector $[\mathbf{u}]_{\mathcal{B}}$.

(c) Find $T(\mathbf{u})$ for the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given that

$$T(\mathbf{v}_1) = \begin{bmatrix}3\\4\end{bmatrix}, \quad T(\mathbf{v}_2) = \begin{bmatrix}2\\3\end{bmatrix}, \quad T(\mathbf{v}_3) = \begin{bmatrix}-1\\7\end{bmatrix}.$$

Answers:

(a) It is sufficient to show that $\text{RREF}([\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3]) = I_3$.

(b) $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix}3\\4\\-2\end{bmatrix}$

(c) $T(\mathbf{u}) = \begin{bmatrix}19\\10\end{bmatrix}$

20. (a) Find the matrix associated to the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects a vector through the line $y = x$, followed by a counterclockwise rotation of $60°$.

(b) Find the vector obtained if we apply the transformation described in part (a) to vector

$$\mathbf{v} = \begin{bmatrix}3\\2\end{bmatrix}.$$

Answers:

(a) $\begin{bmatrix}-\sqrt{3}/2 & 1/2\\1/2 & \sqrt{3}/2\end{bmatrix}$

(b) $\begin{bmatrix} (-3\sqrt{3}+2)/2 \\ (3+2\sqrt{3})/2 \end{bmatrix} \approx \begin{bmatrix} -1.598 \\ 3.232 \end{bmatrix}$

21. Evaluate the following. Give your answer in the form $a + bi$.

(a) $\dfrac{1+3i}{2+5i}$ (b) $(1+2i)^{10}$

Answers: (a) $\frac{17}{29} + \frac{i}{29}$ (b) $237 - 3116i$

22. Find all complex numbers $z = a + bi$ that satisfy $z^3 = 27i$.

Answers: $\frac{3\sqrt{3}}{2} + \frac{3}{2}i$, $\frac{-3\sqrt{3}}{2} + \frac{3}{2}i$, $-3i$.

23. Perform the following operations. Give your answers in phasor form.

(a) $(3i)(2\angle 25^\circ)^4$

(b) $\dfrac{(10\angle 112^\circ)(8\angle 228^\circ)}{(4\angle 48^\circ)(5\angle 72^\circ)}$

(c) $3\angle 115.2^\circ + 5\angle 195.6^\circ$

Answers: (a) $48\angle 190^\circ$ (b) $4\angle 220^\circ$ (c) $6.245\angle 167.3^\circ$

24. Use the cofactor method to find the inverse of the following matrix.

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 0 & 3 & 1 \\ 6 & -2 & 5 \end{bmatrix}$$

Answer:

$$A^{-1} = \frac{1}{76}\begin{bmatrix} 17 & -18 & 7 \\ 6 & 16 & -2 \\ -18 & 28 & 6 \end{bmatrix}$$

25. Use Cramer's rule to solve the following system.

$$\begin{cases} 2x + 3y - 5z = 2 \\ 3x - y + 2z = 1 \\ 5x + 4y - 6z = 3 \end{cases}$$

Answer: $x = 3/5$, $y = -12/5$, $z = -8/5$

26. Find the eigenvalues and the corresponding eigenspaces of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Is the matrix diagonalizable?

Answers: The eigenvalues are: $\lambda_1 = 0$ (algebraic multiplicity 2) and $\lambda_2 = 3$. The eigenspaces are

$$E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$E_{\lambda_2} = \text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$$

The geometric multiplicity of λ_1 is 2. Since the geometric and algebraic multiplicities of each eigenvalues are the same, we can conclude that the matrix is diagonalizable.

27. Consider the matrix

$$A = \begin{bmatrix}1 & 2\\4 & 3\end{bmatrix}.$$

(a) Find an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

(b) Evaluate A^{10}.

Answers:

$$P = \begin{bmatrix}1 & -1\\2 & 1\end{bmatrix} \quad D = \begin{bmatrix}5 & 0\\0 & -1\end{bmatrix} \quad P^{-1} = \begin{bmatrix}1/3 & 1/3\\-2/3 & 1/3\end{bmatrix}$$

$$A^{10} = \begin{bmatrix}3\,255\,209 & 3\,255\,208\\6\,510\,416 & 6\,510\,417\end{bmatrix}$$

28. Find the eigenvalues of the rotation matrix

$$R(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}.$$

Answers: $e^{i\theta}$ and $e^{-i\theta}$

29. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = [1, 4, 5, 6], \quad \mathbf{v}_2 = [3, -2, 1, 4], \quad \mathbf{v}_3 = [-1, 0, -1, -2].$$

(a) Find a basis of $W^\perp$.

(b) Find an orthonormal basis of $W^\perp$.

Answers: (a) A basis of $W^\perp$ is $\{[-1, -1, 1, 0],\ [-2, -1, 0, 1]\}$.

(b) An orthonormal basis of $W^\perp$ is $\left\{\frac{1}{\sqrt{3}}[-1, -1, 1, 0],\ \frac{1}{\sqrt{3}}[-1, 0, -1, 1]\right\}$

30. Find $\text{proj}_W(\mathbf{v})$ if

$$W = \text{span}\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\0\end{bmatrix}\right\} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix}1\\2\\4\\1\end{bmatrix}.$$

Answer: $\text{proj}_W(\mathbf{v}) = \begin{bmatrix}43/26\\27/13\\97/26\\8/13\end{bmatrix} \approx \begin{bmatrix}1.6538\\2.0769\\3.7308\\0.6154\end{bmatrix}$

31. Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

by finding an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^T$.
Answer:

$$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

32. Find the least squares solution of the following system.

$$\begin{cases} x + y = -3 \\ 2x + 3y = -1 \\ -3x + 2y = 2 \end{cases}$$

Answer: $x = -152/195$ and $y = -17/195$

33. Find the least squares straight line fit to the four points: $(0, 1)$, $(1, 3)$, $(2, 4)$, and $(3, 4)$.
Answer: $y = 1.5 + x$