The Fundamental Theorem of Invertible Matrices

Let $A$ be an $n \times n$ matrix. The following statements are equivalent:

a. $A$ is invertible.

b. $Ax = b$ has a unique solution for every $b$ in $\mathbb{R}^n$.

c. $Ax = 0$ has only the trivial solution.

d. The reduced row echelon form (RREF) of $A$ is $I_n$.

e. $A$ is a product of elementary matrices.

f. $\text{rank}(A) = n$

g. $\text{nullity}(A) = 0$

h. The column vectors of $A$ are linearly independent.

i. The column vectors of $A$ span $\mathbb{R}^n$.

j. The column vectors of $A$ form a basis of $\mathbb{R}^n$.

k. The row vectors of $A$ are linearly independent.

l. The row vectors of $A$ span $\mathbb{R}^n$.

m. The row vectors of $A$ form a basis of $\mathbb{R}^n$.

n. $\det A \neq 0$

o. $0$ is not an eigenvalue of $A$. 