A proof that π is irrational

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The first rigorous proof that π is irrational is from Johann Heinrich Lambert in 1761. He proved that if $x \neq 0$ is rational, then $\tan x$ must be irrational. Since $\tan \frac{\pi}{4} = 1$ is rational, then π must be irrational. The simpler proof given here is due to Ivan Niven in 1947. It only assumes a knowledge of basic Calculus.

We will prove that π^2 is irrational. This stronger result implies that π is irrational. The proof uses the function

$$f(x) = \frac{x^n(1-x)^n}{n!}.$$

Lemma. *For any integer $n \geq 1$,*

(i) *$f(x)$ is a polynomial of the form $f(x) = \frac{1}{n!}\sum_{i=n}^{2n} c_i x^i$ and all the coefficients c_i are integers.*

(ii) *For $0 < x < 1$, we have $0 < f(x) < \frac{1}{n!}$.*

(iii) *The derivatives $f^{(k)}(0)$ and $f^{(k)}(1)$ are integers for all $k \geq 0$.*

Proof. By expanding the binomial $(1-x)^n$ and multiplying each term by x^n, we get the polynomial

$$x^n(1-x)^n = x^n - \binom{n}{1}x^{n+1} + \binom{n}{2}x^{n+2} - \cdots + (-1)^n\binom{n}{n}x^{2n}$$

where all coefficients are integers. It then follows that part (i) holds. To see that part (ii) holds, observe that for any $0 < x < 1$, we have

$$0 < x^n < 1 \quad \text{and} \quad 0 < (1-x)^n < 1 \qquad \Longrightarrow \qquad 0 < \frac{x^n(1-x)^n}{n!} < 1.$$

Let's now show that part (iii) holds. From part (i) it is clear that

$$f^{(k)}(0) = 0, \quad \text{if } k < n \text{ or } k > 2n.$$

For $n \leq k \leq 2n$, we have $f^{(k)}(0) = \frac{k!}{n!}c_k$ which is an integer. Since

$$f(x) = f(1-x),$$

we have

$$f^{(k)}(x) = (-1)^k f^{(k)}(1-x).$$

Therefore, $f^{(k)}(1) = (-1)^k f^{(k)}(0)$ is also an integer for all k. ☐

Theorem. π^2 *is irrational.*

Proof. Assume that π^2 is rational, i.e., $\pi^2 = a/b$ for two positive integers a and b. Let

$$F(x) = b^n \left(\pi^{2n} f(x) - \pi^{2n-2} f^{(2)}(x) + \pi^{2n-4} f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)\right). \tag{1}$$

Observe that for all $0 \le k \le n$,

$$b^n \pi^{2n-2k} = b^n (\pi^2)^{n-k} = b^n \left(\frac{a}{b}\right)^{n-k} = a^{n-k} b^k$$

is an integer. Since $f^{(k)}(0)$ and $f^{(k)}(1)$ are integers, we see that $F(0)$ and $F(1)$ are integers. Differentiating F twice gives

$$F''(x) = b^n \left(\pi^{2n} f^{(2)}(x) - \pi^{2n-2} f^{(4)}(x) + \pi^{2n-4} f^{(6)}(x) - \cdots + (-1)^n f^{(2n+2)}(x)\right). \tag{2}$$

Observe that $f^{(2n+2)}(x) = 0$. From (1) and (2), we get

$$F''(x) + \pi^2 F(x) = b^n \pi^{2n+2} f(x) = \pi^2 a^n f(x). \tag{3}$$

By differentiation, we get

$$\begin{aligned}
\frac{d}{dx}\left(F'(x)\sin \pi x - \pi F(x)\cos \pi x\right) &= \cancel{\pi F'(x)\cos \pi x} + F''(x)\sin \pi x - \cancel{\pi F'(x)\cos \pi x} + \pi^2 F(x)\sin \pi x \\
&= \left(F''(x) + \pi^2 F(x)\right)\sin \pi x \\
&= \pi^2 a^n f(x)\sin \pi x, \quad \text{from (3)}.
\end{aligned}$$

From the Fundamental Theorem of Calculus, we get

$$\begin{aligned}
\pi^2 a^n \int_0^1 f(x)\sin \pi x \, dx &= \left(F'(x)\sin \pi x - \pi F(x)\cos \pi x\right)\Big|_0^1 \\
&= F'(1)\sin \pi - \pi F(1)\cos \pi - F'(0)\sin 0 + \pi F(0)\cos 0 \\
&= \pi\left(F(1) + F(0)\right).
\end{aligned}$$

Thus,

$$\pi a^n \int_0^1 f(x)\sin \pi x \, dx = F(1) + F(0)$$

is an integer. Since $0 < f(x) < 1/n!$ for $0 < x < 1$, then

$$0 < f(x)\sin \pi x < \frac{1}{n!}, \quad \text{for } 0 < x < 1.$$

Therefore, for any integer $n \ge 1$ we have

$$0 < \pi a^n \int_0^1 f(x)\sin \pi x \, dx < \frac{\pi a^n}{n!}.$$

Since for any number a, we have

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$

we can choose n large enough so that $\frac{\pi a^n}{n!} < 1$. This gives us

$$0 < \pi a^n \int_0^1 f(x)\sin \pi x \, dx < 1$$

which is a contradiction since $\pi a^n \int_0^1 f(x)\sin \pi x \, dx$ is an integer. Therefore π^2 is irrational. $\square$