Harmonic Series

The **harmonic series** is the infinite series

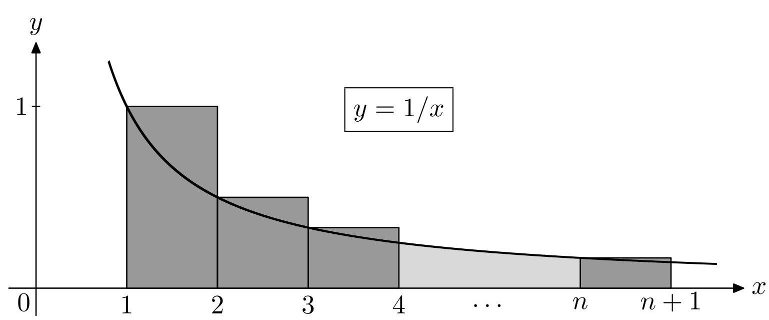
$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

The series diverges to infinity even though $\lim_{n\to\infty} 1/n = 0$. The divergence of the harmonic series was first proved by the medieval French scholar Nicolas d'Oresme (1323–1382). His proof is the following.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\ &\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots\end{aligned}$$

Since the series $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$ diverges, the harmonic series diverges as well.

We can also prove that the harmonic series diverges using integration.



The sum of the area of the shaded rectangles is larger than the area between the x-axis and the graph of $y = 1/x$ over $1 \leq x \leq n+1$. Therefore,

$$\ln(n+1) = \int_1^{n+1} \frac{1}{x}\, dx \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Since

$$\lim_{n\to\infty} \ln(n+1) = \infty,$$

we conclude that the harmonic series diverges.

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