Perimeter of an ellipse

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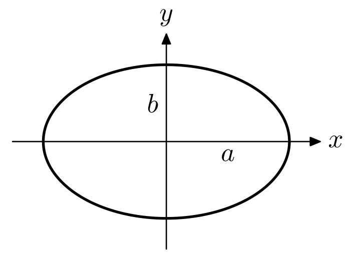
Consider an ellipse of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where we assume that $0 < b < a$.



It is easy to derive that the area of the ellipse is given by $A = \pi ab$. Unfortunately there is no simple formula for the perimeter of an ellipse. We will show that it can be evaluated using an infinite series.

The ellipse can be represented by the parametric equations

$$x = a\cos\theta \quad \text{and} \quad y = b\sin\theta, \quad 0 \le \theta \le 2\pi.$$

To find the perimeter of the ellipse, it is sufficient to find the perimeter in the first quadrant and multiply it by four. The perimeter of the ellipse is then given by

$$\begin{aligned} p &= 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\ &= 4\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta \\ &= 4\int_0^{\pi/2} \sqrt{a^2(1-\cos^2\theta) + b^2\cos^2\theta}\, d\theta \\ &= 4\int_0^{\pi/2} \sqrt{a^2 - (a^2-b^2)\cos^2\theta}\, d\theta \\ &= 4a\int_0^{\pi/2} \sqrt{1 - \varepsilon^2\cos^2\theta}\, d\theta. \end{aligned}$$

Where $\varepsilon = \sqrt{1 - b^2/a^2}$ is the eccentricity of the ellipse. This integral is called an *elliptic integral* and it can't be evaluated using elementary functions.

From the binomial formula

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots, \quad -1 < x < 1$$

we can derive by letting $k = 1/2$ that

$$\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1\cdot 3\cdot 5\cdots(2n-3)x^n}{2^n n!}.$$

By setting $x = -\varepsilon^2 \cos^2\theta$, we deduce that

$$\sqrt{1-\varepsilon^2\cos^2\theta} = 1 - \frac{\varepsilon^2\cos^2\theta}{2} - \sum_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-3)\varepsilon^{2n}\cos^{2n}\theta}{2^n n!}.$$

Note that this series converges for all θ since $0 \le \varepsilon^2\cos^2\theta < 1$. By integrating termwise this series, we can find the perimeter of the ellipse.

$$\begin{aligned} p &= 4a\int_0^{\pi/2}\left(1 - \frac{\varepsilon^2\cos^2\theta}{2} - \sum_{n=2}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-3)\varepsilon^{2n}\cos^{2n}\theta}{2^n n!}\right)d\theta \\ &= 4a\left(\int_0^{\pi/2}d\theta - \frac{\varepsilon^2}{2}\int_0^{\pi/2}\cos^2\theta\,d\theta - \sum_{n=2}^{\infty}\left[\frac{1\cdot 3\cdot 5\cdots(2n-3)\varepsilon^{2n}}{2^n n!}\int_0^{\pi/2}\cos^{2n}\theta\,d\theta\right]\right). \end{aligned}$$

These cosines integrals can be evaluated using Wallis' formula

$$\int_0^{\pi/2}\cos^{2n}\theta\,d\theta = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\cdot\frac{\pi}{2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}\cdot\frac{\pi}{2}, \quad n = 1, 2, 3, \ldots$$

By substitution in the above expression for p, we deduce

$$\begin{aligned} p &= 4a\left(\frac{\pi}{2} - \frac{\varepsilon^2}{2}\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) - \sum_{n=2}^{\infty}\left[\frac{(1\cdot 3\cdot 5\cdots(2n-3)(2n-1)\cdot\varepsilon^n)^2}{(2^n n!)^2(2n-1)}\cdot\frac{\pi}{2}\right]\right) \\ &= 2\pi a\left(1 - \left(\frac{1}{2}\right)^2\varepsilon^2 - \sum_{n=2}^{\infty}\left[\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right)^2\frac{\varepsilon^{2n}}{2n-1}\right]\right) \\ &= 2\pi a\left(1 - \sum_{n=1}^{\infty}\left[\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right)^2\frac{\varepsilon^{2n}}{2n-1}\right]\right). \end{aligned}$$

More explicitly, we obtain the following formula.

$$\boxed{p = 2\pi a\left(1 - \left(\frac{1}{2}\right)^2\frac{\varepsilon^2}{1} - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{\varepsilon^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{\varepsilon^6}{5} - \left(\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\right)^2\frac{\varepsilon^8}{7} - \cdots\right)}$$

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For example, let's use five terms in the above series to approximate the perimeter of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

For this ellipse we have $a = 5$, $b = 4$, and $\varepsilon = \sqrt{1 - b^2/a^2} = 3/5$. We then have

$$\begin{aligned} p &\approx 2\pi a\left(1 - \left(\frac{1}{2}\right)^2\varepsilon^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{\varepsilon^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{\varepsilon^6}{5} - \left(\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\right)^2\frac{\varepsilon^8}{7}\right) \\ &\approx 2\pi(5)\left(1 - \left(\frac{1}{2}\right)^2(3/5)^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{(3/5)^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{(3/5)^6}{5} - \left(\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\right)^2\frac{(3/5)^8}{7}\right) \\ &\approx 28.36. \end{aligned}$$

The ellipse has a perimeter of about 28.36 units.