What is 0^0 ?

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In Calculus we learn that 0^0 is an indeterminate form since knowing that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

is not sufficient information to conclude anything about

$$\lim_{x\to a} f(x)^{g(x)}.$$

For example

$$\lim_{x\to 0^+} x^x = 1 \quad \text{and} \quad \lim_{x\to 0^+} (e^{-1/x})^x = e^{-1}.$$

This might lead us to say that 0^0 should be left undefined.

Following Knuth (see [1] or [2]), I believe that it is a good idea to define $0^0 = 1$ so that we have

$$a^0 = 1 \quad \text{for all } a.$$

If we accept this, we can state that the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

holds for any real numbers x, y, and any nonnegative integer n.

If we leave 0^0 undefined, then the binomial theorem fails to be meaningful for several cases such as $x = 1$, $y = 0$, and $n = 1$. In this case, we would get

$$(1+0)^1 = \binom{1}{0} 1^0 0^1 + \binom{1}{1} 1^1 0^0. \tag{1}$$

The only way to make equation (1) valid is to define $0^0 = 1$.

References

[1] Graham, Knuth, Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, Addison-Wesley, (1994)

[2] Knuth, Donald E., *Two notes on notation*, American Mathematical Monthly **99**, (1992), 403-422.