

# Twelve coins problem

We use a decision tree to describe a solution of the following famous problem.

You have a balance scale and twelve coins, one of which is counterfeit. The counterfeit is lighter or heavier than the other coins. Determine the counterfeit in three weighings, and tell if it is heavier or lighter.

Let's label the twelve coins  $x_1, x_2, \dots, x_{12}$ . We use the symbols “=” for *has the same weight*, “>” for *is heavier*, and “<” for *is lighter*.

| First weighing                | Second weighing            | Third weighing                               |
|-------------------------------|----------------------------|----------------------------------------------|
| $x_1x_2x_3x_4 = x_5x_6x_7x_8$ | $x_1x_9 = x_{10}x_{11}$    | $x_1 = x_{12}$ is impossible                 |
|                               |                            | $x_1 > x_{12} \implies x_{12}$ is lighter    |
|                               |                            | $x_1 < x_{12} \implies x_{12}$ is heavier    |
|                               | $x_1x_9 > x_{10}x_{11}$    | $x_{10} = x_{11} \implies x_9$ is heavier    |
|                               |                            | $x_{10} > x_{11} \implies x_{11}$ is lighter |
|                               |                            | $x_{10} < x_{11} \implies x_{10}$ is lighter |
|                               | $x_1x_9 < x_{10}x_{11}$    | $x_{10} = x_{11} \implies x_9$ is lighter    |
|                               |                            | $x_{10} > x_{11} \implies x_{10}$ is heavier |
|                               |                            | $x_{10} < x_{11} \implies x_{11}$ is heavier |
| $x_1x_2x_3x_4 > x_5x_6x_7x_8$ | $x_1x_2x_5 = x_3x_6x_{12}$ | $x_7 = x_8 \implies x_4$ is heavier          |
|                               |                            | $x_7 > x_8 \implies x_8$ is lighter          |
|                               |                            | $x_7 < x_8 \implies x_7$ is lighter          |
|                               | $x_1x_2x_5 > x_3x_6x_{12}$ | $x_1 = x_2 \implies x_6$ is lighter          |
|                               |                            | $x_1 > x_2 \implies x_1$ is heavier          |
|                               |                            | $x_1 < x_2 \implies x_2$ is heavier          |
|                               | $x_1x_2x_5 < x_3x_6x_{12}$ | $x_3 = x_{12} \implies x_5$ is lighter       |
|                               |                            | $x_3 > x_{12} \implies x_3$ is heavier       |
|                               |                            | $x_3 < x_{12}$ is impossible                 |
| $x_1x_2x_3x_4 < x_5x_6x_7x_8$ | $x_1x_2x_5 = x_3x_6x_{12}$ | $x_7 = x_8 \implies x_4$ is lighter          |
|                               |                            | $x_7 > x_8 \implies x_7$ is heavier          |
|                               |                            | $x_7 < x_8 \implies x_8$ is heavier          |
|                               | $x_1x_2x_5 > x_3x_6x_{12}$ | $x_3 = x_{12} \implies x_5$ is heavier       |
|                               |                            | $x_3 > x_{12}$ is impossible                 |
|                               |                            | $x_3 < x_{12} \implies x_3$ is lighter       |
|                               | $x_1x_2x_5 < x_3x_6x_{12}$ | $x_1 = x_2 \implies x_6$ is heavier          |
|                               |                            | $x_1 > x_2 \implies x_2$ is lighter          |
|                               |                            | $x_1 < x_2 \implies x_1$ is lighter          |