

Sieve of Eratosthenes

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The **sieve of Eratosthenes** is a simple method to find all prime numbers up to a specified integer.

Prime Numbers

Definition

A positive integer greater than 1 is **prime** if it is only divisible by 1 and itself – otherwise it is **composite**. The number 1 is neither prime nor composite.

Theorem (Fundamental Theorem of Arithmetic)

Every positive integer greater than 1 is either a prime or can be uniquely expressed as a product of primes.

Prime Numbers

Theorem

If n is composite, then n has a prime divisor less than or equal to \sqrt{n} .

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Proof.

- Let $n = ab$, $1 < a < n$, $1 < b < n$.
- We can't have both $a > \sqrt{n}$ and $b > \sqrt{n}$ since this would lead to $ab > n$.
- Therefore, n must have a prime divisor less than or equal to \sqrt{n} .



Sieve of Eratosthenes

Let's find all primes less or equal to 100. Since $\sqrt{100} = 10$, any composite integer between 1 and 100 will have a prime divisor less or equal to 10.

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Consider all positive integers from 1 to 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Sieve of Eratosthenes

Since 1 is not a prime, 2 is the first **prime number**.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
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Sieve of Eratosthenes

Now let's cross all multiples of 2.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
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Since 3 is not crossed, it is a **prime number**.

Sieve of Eratosthenes

Let's cross all multiples of 3.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
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Since 5 is not crossed, it is a **prime number**.

Sieve of Eratosthenes

Let's cross all multiples of 5.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
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Sieve of Eratosthenes

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Since 7 is not crossed, it is a **prime number**.

Sieve of Eratosthenes

Let's cross all multiples of 7.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
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91	92	93	94	95	96	97	98	99	100

Since 11 is not crossed, it is a **prime number**.

Sieve of Eratosthenes

Since $11 > 10 = \sqrt{100}$, all numbers not crossed are primes.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
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We have found all **prime numbers** up to 100.