Methods of Proof Examples

Gilles Cazelaïs

We start with an example of a direct proof.

Proposition 1. If \( n \) is an odd integer, then \( n^2 \) is an odd integer.

Proof. Let \( n \) be an odd integer, then \( n = 2k + 1 \) for some integer \( k \). Then,

\[
    n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.
\]

Since \( n^2 = 2m + 1 \) for the integer \( m = 2k^2 + 2k \), we conclude that \( n^2 \) is odd. \( \square \)

To prove a proposition in the form \( p \rightarrow q \), it is sufficient to prove its logically equivalent contrapositive \( \neg q \rightarrow \neg p \). This is called a proof by contrapositive.

Proposition 2. If \( n^2 \) is an even integer, then \( n \) is an even integer.

Proof. The contrapositive of

\[
    (n^2 \text{ is an even integer}) \rightarrow (n \text{ is an even integer})
\]

is

\[
    \neg(n \text{ is an even integer}) \rightarrow \neg(n^2 \text{ is an even integer})
\]

which is equivalent to

\[
    (n \text{ is an odd integer}) \rightarrow (n^2 \text{ is an odd integer})
\]

which was proved in Proposition 1. \( \square \)

A proof by contradiction works as follows. To prove \( p \), start by assuming that \( p \) is false and deduce consequences. If you deduce a contradiction with something known or assumed to be true, then the initial assumption that \( p \) is false was wrong. Therefore, \( p \) must be true.
Proposition 3. $\sqrt{2}$ is an irrational number.

Proof. Assume that $\sqrt{2}$ is a rational number. Then, $\sqrt{2} = a/b$ for two positive integers $a$ and $b$. Assume that $a$ and $b$ have no common factors so that the fraction $a/b$ is an irreducible fraction. By squaring both sides of $\sqrt{2} = a/b$, we deduce $2 = a^2/b^2$. Therefore

$$2b^2 = a^2$$

which implies that $a^2$ is even. From Proposition 2, we conclude that $a$ is even, i.e., $a = 2k$ for some integer $k$. Substitute $a = 2k$ in equation (1) to get

$$b^2 = 2k^2.$$  

We conclude that $b^2$ is even which implies that $b$ is even. We have derived that both $a$ and $b$ are even but this a contradiction since we assumed that the fraction $a/b$ was irreducible. Therefore, $\sqrt{2}$ is an irrational number. \qed

Let’s now look an example of a proof by cases.

Proposition 4. There exists irrational numbers $a$ and $b$ such that $a^b$ is rational.

Proof. Consider the number $\sqrt{2}^{\sqrt{2}}$ which is either rational or irrational.

Case 1. If $\sqrt{2}^{\sqrt{2}}$ is rational, by choosing $a = b = \sqrt{2}$ we get that $a^b$ is rational.

Case 2. If $\sqrt{2}^{\sqrt{2}}$ is irrational, by choosing

$$a = \sqrt{2}^{\sqrt{2}} \text{ and } b = \sqrt{2}$$

we get that $a^b$ is rational since

$$a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2.$$ \qed

Observe that this proof does not tell us whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. We used the fact that it is either rational or irrational.