Fibonacci Sequence and the Golden Ratio

Gilles Cazelais

The Fibonacci sequence is the sequence defined by

\[
F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad \text{for } n = 2, 3, 4, \ldots
\]

The first few terms of the Fibonacci sequence are the following.

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>F_n</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td>55</td>
<td>89</td>
<td>144</td>
<td>233</td>
<td>377</td>
</tr>
</tbody>
</table>

The number \(\phi = (1 + \sqrt{5})/2 = 1.61803\ldots\) is called the golden ratio. It satisfies the property

\[
\phi = 1 + \frac{1}{\phi}. \quad (1)
\]

We will prove that the sequence of ratio of successive Fibonacci numbers \(F_{n+1}/F_n\) converges to the golden ratio.

\[
\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi
\]

Observe that \(F_{14}/F_{13} = 377/233 \approx 1.618025\) is already pretty close to \(\phi\).

Proof. Consider the sequence \(R_n = F_{n+1}/F_n\), for \(n = 1, 2, 3, \ldots\). By the definition of Fibonacci numbers, we get

\[
R_n = \frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{1}{R_{n-1}}. \quad (2)
\]

From (1) and (2) we can deduce that for all \(n = 1, 2, 3, \ldots\), we have

\[
|R_n - \phi| = \left| (1 + \frac{1}{R_{n-1}}) - \left( 1 + \frac{1}{\phi} \right) \right| \\
= \left| \frac{1}{R_{n-1}} - \frac{1}{\phi} \right| \\
= \left| \frac{\phi - R_{n-1}}{R_{n-1}\phi} \right| \\
\leq \frac{1}{\phi} |R_{n-1} - \phi| \\
\leq \left( \frac{1}{\phi} \right)^{n-1} |R_1 - \phi|.
\]

Since \(0 < 1/\phi < 1\), then

\[
\lim_{n \to \infty} \left( \frac{1}{\phi} \right)^{n-1} = 0 \implies \lim_{n \to \infty} |R_n - \phi| = 0.
\]

Therefore,

\[
\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi.
\]

QED