Fibonacci Sequence and the Golden Ratio

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The Fibonacci sequence is the sequence defined by

$$
\begin{aligned}
F_0 &= 0, \\
F_1 &= 1, \\
F_n &= F_{n-1} + F_{n-2}, \quad \text{for } n = 2, 3, 4, \ldots .
\end{aligned}
$$

The first few terms of the Fibonacci sequence are the following.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377

The number $\phi = \left(1 + \sqrt{5}\right)/2 = 1.61803\ldots$ is called the *golden ratio*. It satisfies the property

$$\phi = 1 + \frac{1}{\phi}. \tag{1}$$

We will prove that the sequence of ratio of successive Fibonacci numbers F_{n+1}/F_n converges to the golden ratio.

$$\boxed{\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi}$$

Observe that $F_{14}/F_{13} = 377/233 \approx 1.618025$ is already pretty close to ϕ.

Proof. Consider the sequence $R_n = F_{n+1}/F_n$, for $n = 1, 2, 3, \ldots$. By the definition of Fibonacci numbers, we get

$$R_n = \frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{1}{R_{n-1}}. \tag{2}$$

From (1) and (2) we can deduce that for all $n = 1, 2, 3, \ldots$, we have

$$
\begin{aligned}
|R_n - \phi| &= \left| \left(1 + \frac{1}{R_{n-1}}\right) - \left(1 + \frac{1}{\phi}\right) \right| \\
&= \left| \frac{1}{R_{n-1}} - \frac{1}{\phi} \right| \\
&= \left| \frac{\phi - R_{n-1}}{R_{n-1}\phi} \right| \\
&\leqslant \frac{1}{\phi} |R_{n-1} - \phi| \\
&\leqslant \left(\frac{1}{\phi}\right)^{n-1} |R_1 - \phi| .
\end{aligned}
$$

Since $0 < 1/\phi < 1$, then

$$\lim_{n\to\infty} \left(\frac{1}{\phi}\right)^{n-1} = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} |R_n - \phi| = 0.$$

Therefore,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi.$$

QED