Basic Counting Principles

Multiplication Principle
Consider a multistep process in which

Step 1 has \( n_1 \) possible outcomes,

Step 2 has \( n_2 \) possible outcomes,

\[ \cdots \]

Step \( r \) has \( n_r \) possible outcomes.

Then, the entire process has \( n_1 \times n_2 \times \cdots \times n_r \) possible outcomes.

Inclusion-Exclusion Principle
If \( A \) and \( B \) are two finite sets, then

\[ |A \cup B| = |A| + |B| - |A \cap B| . \]

A permutation of \( n \) distinct items taken \( r \) at a time is an ordered list of \( r \) distinct items chosen from a set of \( n \) distinct items. The number of such permutations is given by

\[ P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1) = \frac{n!}{(n - r)!} . \]

Observe that \( P(n, n) = n! \).

A combination of \( n \) distinct items taken \( r \) at a time is an unordered set of \( r \) distinct items chosen from a set of \( n \) distinct items. The number of such combinations is given by

\[ C(n, r) = \frac{n!}{r!(n - r)!} . \]

Another notation for \( C(n, r) \) is \( ^n \! \text{C}_r \).

Problems

1. (a) How many bit strings of length 8 are possible?
   (b) How many of these start with a 1 or end with 00?
   (c) How many have at most two 0s?
   \textbf{Answers:} (a) 256 (b) 160 (c) 37

2. How many functions are there from a set with 6 elements to a set with 4 elements.
   \textbf{Answer:} 4096

3. How many one-to-one functions are there from a set with 6 elements to a set with
   (a) 4 elements  (b) 6 elements  (c) 10 elements.
   \textbf{Answers:} (a) 0 (b) 720 (c) 151 200

4. Each user on a computer system has a password, which is six to eight characters long, where each character is a letter (case sensitive) or a digit. If each password must contain at least one digit and at least one letter, how many possible passwords are there?
   \textbf{Answer:} 167 410 838 583 040 \approx 1.67 \times 10^{14}

5. (a) How many different 7-place license plates are possible if the first 3 places are for capital letters and the other 4 for digits?
   (b) What if no letter and digit can be repeated in a single license plate?
   (c) What if adjacent letters and digits have to be different?
   \textbf{Answers:} (a) 175 760 000 (b) 78 624 000 (c) 118 462 500
6. The 10 letters ABCDEFGHIJ are used to form strings of length 7 (order matters).
   (a) How many possible strings are there if we do not allow repetition?
   (b) How many strings begin with the letter G if repetitions are allowed?
   (c) How many strings contain the substring GAB if repetitions are not allowed?
   (d) How many strings begin or end with the substring GAB if repetitions is allowed?
   (e) How many strings contain the letters A and B, with A somewhere to the left of B, if repetitions are not allowed?

   Answers: (a) 604800 (b) 1 000 000 (c) 4200 (d) 19990 (e) 14120

7. How many different subsets of 5 letters are possible from the 26 letters of the alphabet?

   Answer: 65780

8. In how many ways can we select a chairperson, secretary, and treasurer from a group of 25 persons?

   Answer: 13800

9. How many strings of five decimal digits
   (a) do not contain the same digit five times?
   (b) begin with an odd digit?
   (c) have exactly three digits that are 5s?

   Answers: (a) 99 990 (b) 50 000 (c) 810

10. A biologist is attempting to classify 20,000 species of insects by assigning 3 letter initials (not necessarily distinct) to each species. Is it possible to classify all the species in this way? Answer: No

11. We have 25 lightbulbs distributed as follows and we select 5 at random (order does not matter).

<table>
<thead>
<tr>
<th>40-W</th>
<th>60-W</th>
<th>75-W</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>6</td>
<td>10</td>
</tr>
</tbody>
</table>

   (a) In how many ways can this be done.
   (b) How many selections will contain exactly two 75-W bulbs.
   (c) How many selections will contain at least one 75-W bulbs.
   (d) How many selections will contain two 40-W and three 60-W bulbs

   Answers: (a) 53130 (b) 20475 (c) 50127 (d) 720

12. How many 5-card hands are possible from a standard deck of 52 cards? How many of these hands will have exactly three hearts and two spades? How many hands will have 3 kings and a pair?

   Answers: 2598960, 22308, 288

13. Consider the following 10 × 10 grid.

   (a) How many paths joining the two points A and C are possible if we start at point A and are only allowed to travel right and up on the grid?
   (b) Of all the paths from A to C found in (a), how many pass through point B?

   Answers: (a) 184756 (b) 58212

14. How many 10-bit strings contain five consecutive zeros or five consecutive ones?

   Answer: 222

15. How many different ways are there to choose a dozen donuts from the five varieties at a donut shop?

   Answer: 1820

16. The number 5 can be expressed as a sum of 3 positive integers, taking order into account, in 6 ways, namely

   \[ 1 + 1 + 3 = 1 + 3 + 1 = 3 + 1 + 1 = 1 + 2 + 2 = 2 + 1 + 2 = 2 + 2 + 1. \]

   In how many ways can 10 be expressed as a sum of 5 positive integers, taking order into account?

   Answer: 126