

Basic Counting Principles

Multiplication Principle

Consider a multistep process in which

Step 1 has n_1 possible outcomes,

Step 2 has n_2 possible outcomes,

...

Step r has n_r possible outcomes.

Then, the entire process has $n_1 \times n_2 \times \cdots \times n_r$ possible outcomes.

Inclusion-Exclusion Principle

If A and B are two finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

A **permutation of n distinct items taken r at a time** is an *ordered* list of r distinct items chosen from a set of n distinct items. The number of such permutations is given by

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

Observe that $P(n, n) = n!$.

A **combination of n distinct items taken r at a time** is an *unordered* set of r distinct items chosen from a set of n distinct items. The number of such combinations is given by

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

Another notation for $C(n, r)$ is $\binom{n}{r}$.

Problems

- (a) How many bit strings of length 8 are possible?
(b) How many of these start with a 1 or end with 00?
(c) How many have at most two 0s?

Answers: (a) 256 (b) 160 (c) 37

- How many functions are there from a set with 6 elements to a set with 4 elements.

Answer: 4096

- How many one-to-one functions are there from a set with 6 elements to a set with
(a) 4 elements (b) 6 elements (c) 10 elements.

Answers: (a) 0 (b) 720 (c) 151 200

- Each user on a computer system has a password, which is six to eight characters long, where each character is a letter (case sensitive) or a digit. If each password must contain at least one digit and at least one letter, how many possible passwords are there?

Answer: $167\,410\,838\,583\,040 \approx 1.67 \times 10^{14}$

- (a) How many different 7-place license plates are possible if the first 3 places are for capital letters and the other 4 for digits?
(b) What if no letter and digit can be repeated in a single license plate?
(c) What if adjacent letters and digits have to be different?

Answers: (a) 175 760 000 (b) 78 624 000 (c) 118 462 500

6. The 10 letters ABCDEFGHIJ are used to form strings of length 7 (order matters).
- How many possible strings are there if we do not allow repetition?
 - How many strings begin with the letter G if repetitions are allowed?
 - How many strings contain the substring GAB if repetitions are not allowed?
 - How many strings begin or end with the substring GAB if repetitions is allowed?
 - How many strings contain the letters A and B, with A somewhere to the left of B, if repetitions are not allowed?

Answers: (a) 604800 (b) 1 000 000 (c) 4200 (d) 19990 (e) 141120

7. How many different subsets of 5 letters are possible from the the 26 letters of the alphabet?

Answer: 65780

8. In how many ways can we select a chairperson, secretary, and treasurer form a group of 25 persons?

Answer: 13800

9. How many strings of five decimal digits

- do not contain the same digit five times?
- begin with an odd digit?
- have exactly three digits that are 5s?

Answers: (a) 99 990 (b) 50 000 (c) 810

10. A biologist is attempting to classify 20,000 species of insects by assigning 3 letter initials (not necessarily distinct) to each species. Is it possible to classify all the species in this way? *Answer:* No

11. We have 25 lightbulbs distributed as follows and we select 5 at random (order does not matter).

| | | |
|------|------|------|
| 40-W | 60-W | 75-W |
| 9 | 6 | 10 |

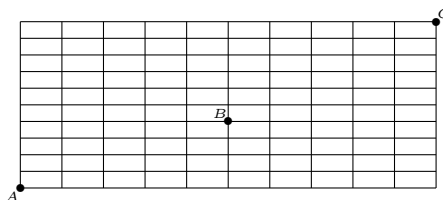
- In how many ways can this be done.
- How many selections will contain exactly two 75-W bulbs.
- How many selections will contain at least one 75-W bulbs.
- How many selections will contain two 40-W and three 60-W bulbs

Answers: (a) 53130 (b) 20475 (c) 50127 (d) 720

12. How many 5-card hands are possible from a standard deck of 52 cards? How many of these hands will have exactly three hearts and two spades? How many hands will have 3 kings and a pair?

Answers: 2598960, 22308, 288

13. Consider the following 10×10 grid.



- How many paths joining the two points A and C are possible if we start at point A and are only allowed to travel right and up on the grid?
- Of all the paths from A to C found in (a), how many pass through point B ?

Answers: (a) 184756 (b) 58212

14. How many 10-bit strings contain five consecutive zeros or five consecutive ones?

Answer: 222

15. How many different ways are there to choose a dozen donuts from the five varieties at a donut shop?

Answer: 1820

16. The number 5 can be expressed as a sum of 3 positive integers, taking order into account, in 6 ways, namely

$$1 + 1 + 3 = 1 + 3 + 1 = 3 + 1 + 1 = 1 + 2 + 2 = 2 + 1 + 2 = 2 + 2 + 1.$$

In how many ways can 10 be expressed as a sum of 5 positive integers, taking order into account?

Answer: 126