Binomial Theorem

For any set S with n elements, the number of subsets of S with r elements is given by

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}.$$

A useful combinatorial formula is Pascal's identity

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}, \quad 1 \leqslant r \leqslant n.$$

It can be proved by the following combinatorial argument. Consider a set S with $n+1$ elements and fix attention on a particular element in the set, call it element a. There are $\binom{n}{r-1}$ subsets of S with r elements that contain a, and there are $\binom{n}{r}$ subsets of S with r elements that do not contain a. Since there are a total of $\binom{n+1}{r}$ subsets of S with r elements, Pascal's identity holds.

Binomial Theorem. For any $n \in \mathbf{N}$ and any $x, y \in \mathbf{R}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. We prove it by induction. For $n = 1$ we have

$$(x+y)^1 = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0 = x + y.$$

Assume the formula holds for some fixed arbitrary $n \in \mathbf{N}$. Then,

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n \\
&= (x+y) \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\
&= \underbrace{\sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k}}_{\text{set } \ell=k+1} + \underbrace{\sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k}}_{\text{set } \ell=k} \\
&= \sum_{\ell=1}^{n+1} \binom{n}{\ell-1} x^\ell y^{n+1-\ell} + \sum_{\ell=0}^{n} \binom{n}{\ell} x^\ell y^{n+1-\ell} \\
&= \sum_{\ell=1}^{n} \binom{n}{\ell-1} x^\ell y^{n+1-\ell} + x^{n+1} + y^{n+1} + \sum_{\ell=1}^{n} \binom{n}{\ell} x^\ell y^{n+1-\ell} \\
&= x^{n+1} + \sum_{\ell=1}^{n} \left[\binom{n}{\ell-1} + \binom{n}{\ell} \right] x^\ell y^{n+1-\ell} + y^{n+1} \\
&= x^{n+1} + \sum_{\ell=1}^{n} \binom{n+1}{\ell} x^\ell y^{n+1-\ell} + y^{n+1} \\
&= \sum_{\ell=0}^{n+1} \binom{n+1}{\ell} x^\ell y^{n+1-\ell}
\end{aligned}$$

which completes the proof by induction.