Big-O Examples

Definition Let $f$ and $g$ be real-valued functions. We say that $f(x)$ is $O(g(x))$ if there are constants $C$ and $k$ such that

$$|f(x)| \leq C|g(x)| \quad \text{for all } x > k.$$  

Example 1 Show that $f(x) = 4x^2 - 5x + 3$ is $O(x^2)$.

$$|f(x)| = |4x^2 - 5x + 3|$$

$$\leq |4x^2| + |-5x| + |3|$$

$$\leq 4x^2 + 5x + 3, \quad \text{for all } x > 0$$

$$\leq 4x^2 + 5x + 3, \quad \text{for all } x > 1$$

$$\leq 12x^2, \quad \text{for all } x > 1$$

We conclude that $f(x)$ is $O(x^2)$. Observe that $C = 12$ and $k = 1$ from the definition of big-O.

Example 2 Show that $f(x) = (x + 5) \log_2(3x^2 + 7)$ is $O(x \log_2 x)$.

$$|f(x)| = |(x + 5) \log_2(3x^2 + 7)|$$

$$= (x + 5) \log_2(3x^2 + 7), \quad \text{for all } x > -5$$

$$\leq (x + 5x) \log_2(3x^2 + 7x^2), \quad \text{for all } x > 1$$

$$\leq 6x \log_2(10x^2), \quad \text{for all } x > 1$$

$$\leq 6x \log_2(x^3), \quad \text{for all } x > 10$$

$$\leq 18x \log_2 x, \quad \text{for all } x > 10$$

We conclude that $f(x)$ is $O(x \log_2 x)$. Observe that $C = 18$ and $k = 10$ from the definition of big-O.

Example 3 Show that $f(x) = (x^2 + 5 \log_2 x)/(2x + 1)$ is $O(x)$.

Since $\log_2 x < x$ for all $x > 0$, we conclude that

$$5 \log_2 x < 5x < 5x^2, \quad \text{for all } x > 1.$$  

Since $2x + 1 > 2x$, we conclude that

$$\frac{1}{2x + 1} < \frac{1}{2x} \quad \text{for all } x > 0.$$  

Therefore,

$$|f(x)| = \left| \frac{x^2 + 5 \log_2 x}{2x + 1} \right|$$

$$= \frac{x^2 + 5 \log_2 x}{2x + 1}, \quad \text{for all } x > 1$$

$$\leq \frac{x^2}{2x}, \quad \text{for all } x > 1$$

$$\leq 3x, \quad \text{for all } x > 1$$

We conclude that $f(x)$ is $O(x)$. Observe that $C = 3$ and $k = 1$ from the definition of big-O.