The Trapezoidal Rule

For definite integrals such as

$$\int_0^1 \sqrt{1-x^3}\,dx \quad \text{or} \quad \int_0^1 e^{-x^2}\,dx$$

we can't use the Fundamental Theorem of Calculus to evaluate them since there are no elementary functions that are antiderivatives of $\sqrt{1-x^3}$ or e^{-x^2}. The best we can do is to use approximation methods for such integrals.

The trapezoidal rule is a numerical method that approximates the value of a definite integral. We consider the definite integral

$$\int_a^b f(x)\,dx.$$

We assume that $f(x)$ is continuous on $[a, b]$ and we divide $[a, b]$ into n subintervals of equal length

$$\Delta x = \frac{b-a}{n}$$

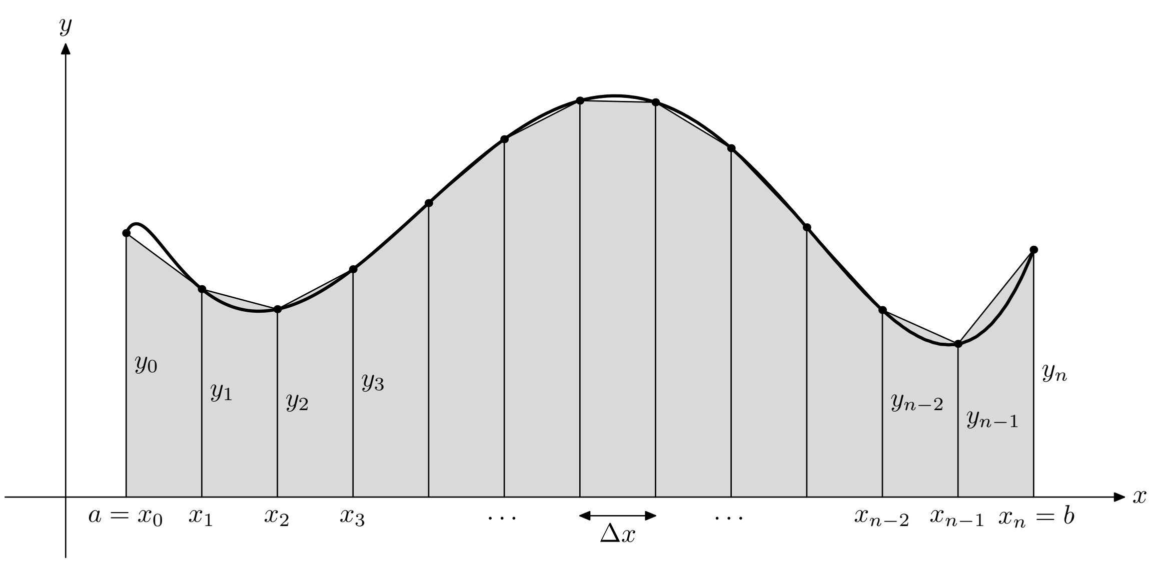
using the $n+1$ points

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \ldots, \quad x_n = a + n\Delta x = b.$$

We can compute the value of $f(x)$ at these points.

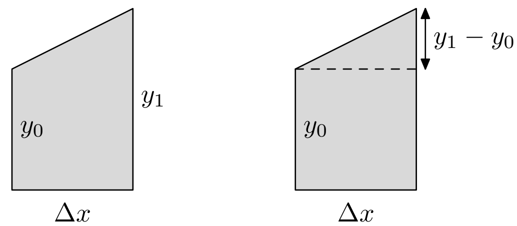
$$y_0 = f(x_0), \quad y_1 = f(x_1), \quad y_2 = f(x_2), \quad \ldots, \quad y_n = f(x_n)$$

We approximate the integral by using n trapezoids formed by using straight line segments between the points (x_{i-1}, y_{i-1}) and (x_i, y_i) for $1 \le i \le n$ as shown in the figure below.



The area of a trapezoid is obtained by adding the area of a rectangle and a triangle.

$$A = y_0\,\Delta x + \frac{1}{2}(y_1 - y_0)\Delta x = \frac{(y_0 + y_1)\Delta x}{2}.$$



By adding the area of the n trapezoids, we obtain the approximation

$$\int_a^b f(x)\,dx \approx \frac{(y_0 + y_1)\Delta x}{2} + \frac{(y_1 + y_2)\Delta x}{2} + \frac{(y_2 + y_3)\Delta x}{2} + \cdots + \frac{(y_{n-1} + y_n)\Delta x}{2}$$

which simplifies to the trapezoidal rule formula.

$$\boxed{\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)}$$

Example 1. Use the trapezoidal rule with $n = 8$ to estimate

$$\int_1^5 \sqrt{1 + x^2}\;dx.$$

Solution. For $n = 8$, we have $\Delta x = \frac{5-1}{8} = 0.5$. We compute the values of $y_0,\ y_1,\ y_2, \ldots,\ y_8$.

x	1	1.5	2	2.5	3	3.5	4	4.5	5
$y = \sqrt{1 + x^2}$	$\sqrt{2}$	$\sqrt{3.25}$	$\sqrt{5}$	$\sqrt{7.25}$	$\sqrt{10}$	$\sqrt{13.25}$	$\sqrt{17}$	$\sqrt{21.25}$	$\sqrt{26}$

Therefore,

$$\begin{aligned}\int_1^5 \sqrt{1 + x^2}\;dx &\approx \frac{0.5}{2}\left(\sqrt{2} + 2\sqrt{3.25} + 2\sqrt{5} + 2\sqrt{7.25} + 2\sqrt{10} + 2\sqrt{13.25} + 2\sqrt{17} + 2\sqrt{21.25} + \sqrt{26}\right)\\ &\approx \boxed{12.76}\end{aligned}$$

Example 2. The following points were found empirically.

x	2.1	2.4	2.7	3.0	3.3	3.6
y	3.2	2.7	2.9	3.5	4.1	5.2

Use the trapezoidal rule to estimate $\displaystyle\int_{2.1}^{3.6} y\,dx$.

Solution. By inspection, we see that $\Delta x = 0.3$. Therefore,

$$\begin{aligned}\int_{2.1}^{3.6} y\,dx &\approx \frac{0.3}{2}\left(3.2 + 2(2.7) + 2(2.9) + 2(3.5) + 2(4.1) + 5.2\right)\\ &\approx \boxed{5.22}\end{aligned}$$