# Integration by Substitution

Suppose that $F(u)$ is an antiderivative of $f(u)$, i.e.,

$$\int f(u)\,du = F(u) + C.$$

Let $u = u(x)$ be a differentiable function. From the chain rule we deduce that

$$\frac{d}{dx}F(u(x)) = F'(u(x))\,u'(x) = f(u(x))\,u'(x).$$

Therefore,

$$\int f(u(x))\,u'(x)\,dx = F(u(x)) + C.$$

We conclude that

$$\boxed{\int f(u(x))\,u'(x)\,dx = \int f(u)\,du, \quad \text{where } u = u(x).}$$

This is the method of substitution for indefinite integrals.

Observe that

$$\int_a^b f(u(x))\,u'(x)\,dx = F(u(x))\Big|_a^b = F(u(b)) - F(u(a)).$$

Therefore,

$$\boxed{\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.}$$

This is the method of substitution for definite integrals.

**Example 1.** Evaluate

$$\int 3x^2(x^3+1)^5\,dx.$$

*Solution.* Let $u = x^3 + 1 \implies du = 3x^2\,dx$. Therefore,

$$\begin{aligned}\int 3x^2(x^3+1)^5\,dx &= \int u^5\,du \\ &= \frac{u^6}{6} + C \\ &= \boxed{\frac{1}{6}\left(x^3+1\right)^6 + C}\end{aligned}$$

**Example 2.** Evaluate

$$\int_0^2 x\sqrt{2x^2+1}\,dx.$$

*Solution.* Let $u = 2x^2 + 1$. Then, $du = 4x\,dx \implies \frac{du}{4} = x\,dx$. We also have

$$\begin{cases} x = 2 \implies u = 2(2)^2 + 1 = 9 \\ x = 0 \implies u = 2(0)^2 + 1 = 1. \end{cases}$$

Therefore,

$$\begin{aligned}\int_0^2 x\sqrt{2x^2+1}\,dx &= \int_1^9 \sqrt{u}\,\frac{du}{4} = \frac{1}{4}\int_1^9 u^{1/2}\,du \\ &= \frac{1}{4}\left(\frac{u^{3/2}}{3/2}\right)\Bigg|_1^9 = \frac{1}{6}\left(9^{3/2} - 1^{3/2}\right) \\ &= \boxed{\frac{13}{3}}\end{aligned}$$