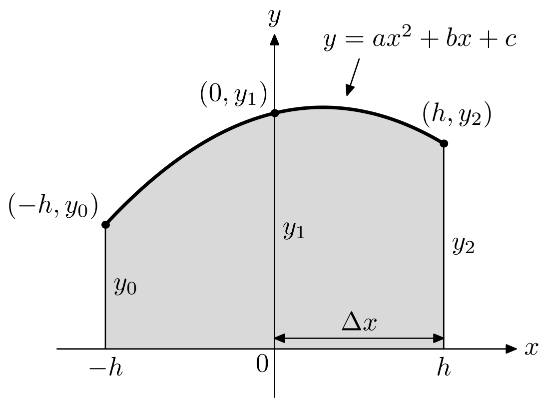
Simpson's Rule

Simpson's rule is a numerical method that approximates the value of a definite integral by using quadratic polynomials.

Let's first derive a formula for the area under a parabola of equation $y = ax^2 + bx + c$ passing through the three points: $(-h, y_0)$, $(0, y_1)$, (h, y_2).



$$
\begin{aligned}
A &= \int_{-h}^{h} (ax^2 + bx + c)\,dx \\
&= \left(\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right)\Bigg|_{-h}^{h} \\
&= \frac{2ah^3}{3} + 2ch \\
&= \frac{h}{3}\left(2ah^2 + 6c\right)
\end{aligned}
$$

Since the points $(-h, y_0)$, $(0, y_1)$, (h, y_2) are on the parabola, they satisfy $y = ax^2 + bx + c$. Therefore,

$$
\begin{aligned}
y_0 &= ah^2 - bh + c \\
y_1 &= c \\
y_2 &= ah^2 + bh + c
\end{aligned}
$$

Observe that

$$y_0 + 4y_1 + y_2 = (ah^2 - bh + c) + 4c + (ah^2 + bh + c) = 2ah^2 + 6c.$$

Therefore, the area under the parabola is

$$A = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right) = \frac{\Delta x}{3}\left(y_0 + 4y_1 + y_2\right).$$

Gilles Cazelais. Typeset with LaTeX on April 23, 2008.

We consider the definite integral

$$\int_a^b f(x)\,dx.$$

We assume that $f(x)$ is continuous on $[a,b]$ and we divide $[a,b]$ into an **even** number n of subintervals of equal length

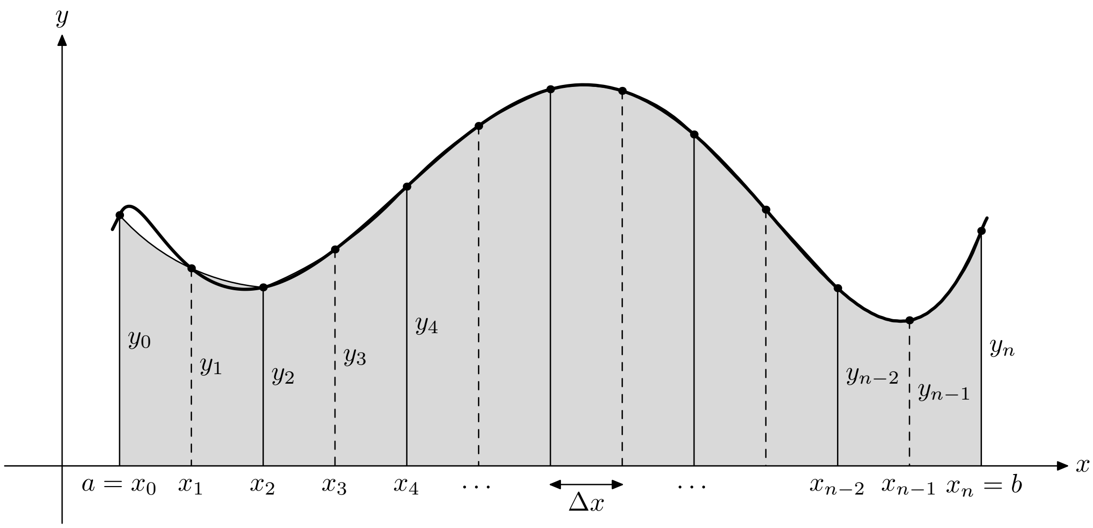
$$\Delta x = \frac{b-a}{n}$$

using the $n+1$ points

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \ldots, \quad x_n = a + n\Delta x = b.$$

We can compute the value of $f(x)$ at these points.

$$y_0 = f(x_0), \quad y_1 = f(x_1), \quad y_2 = f(x_2), \quad \ldots, \quad y_n = f(x_n).$$



We can estimate the integral by adding the areas under the parabolic arcs through three successive points.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + y_2) + \frac{\Delta x}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{\Delta x}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

By simplifying, we obtain Simpson's rule formula.

$$\boxed{\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n)}$$

Example. Use Simpson's rule with $n = 6$ to estimate

$$\int_1^4 \sqrt{1+x^3}\,dx.$$

Solution. For $n = 6$, we have $\Delta x = \frac{4-1}{6} = 0.5$. We compute the values of $y_0,\ y_1,\ y_2, \ldots,\ y_6$.

x	1	1.5	2	2.5	3	3.5	4
$y = \sqrt{1+x^3}$	$\sqrt{2}$	$\sqrt{4.375}$	3	$\sqrt{16.625}$	$\sqrt{28}$	$\sqrt{43.875}$	$\sqrt{65}$

Therefore,

$$\begin{aligned}\int_1^4 \sqrt{1+x^3}\,dx &\approx \frac{0.5}{3}\left(\sqrt{2} + 4\sqrt{4.375} + 2(3) + 4\sqrt{16.625} + 2\sqrt{28} + 4\sqrt{43.875} + \sqrt{65}\right) \\ &\approx \boxed{12.871}\end{aligned}$$