Double Integrals in Polar Coordinates

Consider a region \( R \) described in polar coordinates as
\[
R: \quad \theta_1 \leq \theta \leq \theta_2, \quad r_1 \leq r \leq r_2.
\]

We can use polar coordinates to evaluate a double integral of a function \( f(x, y) \) over region \( R \) as follows.
\[
\iint_{R} f(x, y) \, dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.
\]
Observe that:
\[
x = r \cos \theta, \quad y = r \sin \theta, \quad \text{and} \quad dA = r \, dr \, d\theta
\]

**Example 1.** Evaluate the double integral
\[
\iint_{R} (x + y) \, dA
\]
where \( R \) is the first quadrant region between two circles of radius 1 and 2.

**Solution.** In polar coordinates, the region \( R \) is described as
\[
R: \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 1 \leq r \leq 2.
\]
Therefore,
\[
\iint_{R} (x + y) \, dA = \int_{0}^{\pi/2} \int_{1}^{2} (r \cos \theta + r \sin \theta) \, r \, dr \, d\theta
\]
\[
= \int_{0}^{\pi/2} \int_{1}^{2} r^2 (\cos \theta + \sin \theta) \, dr \, d\theta
\]
\[
= \int_{0}^{\pi/2} \left[ \frac{r^3}{3} \right]_{1}^{2} (\cos \theta + \sin \theta) \, d\theta
\]
\[
\frac{7}{3} \int_{0}^{\pi/2} (\cos \theta + \sin \theta) \, d\theta \\
= \frac{7}{3} \left( \sin \theta - \cos \theta \right)\bigg|_{0}^{\pi/2} \\
= \frac{7}{3} \left( \sin \pi/2 - \cos \pi/2 \right) - \frac{7}{3} (\sin 0 - \cos 0) \\
= \frac{14}{3}
\]

**Example 2.** Evaluate the double integral

\[
\int\int_{R} x^2 \, dA
\]

over the region \( R \) described in rectangular coordinates by

\( R: \quad -3 \leq x \leq 3, \quad 0 \leq y \leq \sqrt{9-x^2} \).

**Solution.** The region \( R \) corresponds to the following half-circle.

In polar coordinates it can be described as

\( R: \quad 0 \leq \theta \leq \pi, \quad 0 \leq r \leq 3 \).

Therefore,

\[
\int\int_{R} x^2 \, dA = \int_{0}^{\pi} \int_{0}^{3} (r \cos \theta)^2 r \, dr \, d\theta \\
= \int_{0}^{\pi} \int_{0}^{3} r^3 \cos^2 \theta \, dr \, d\theta \\
= \int_{0}^{\pi} \frac{r^4}{4} \bigg|_{0}^{3} \cos^2 \theta \, d\theta \\
= \frac{81}{4} \int_{0}^{\pi} \cos^2 \theta \, d\theta \\
= \frac{81}{4} \int_{0}^{\pi} \frac{1 + \cos(2\theta)}{2} \, d\theta \\
= \frac{81}{8} \left( \theta + \frac{\sin(2\theta)}{2} \right)\bigg|_{0}^{\pi} \\
= \frac{81\pi}{8}
\]
Example 3. Find the volume above the $xy$-plane and under the surface $z = 4 - x^2 - y^2$.

Solution. The surface $z = 4 - x^2 - y^2$ is a paraboloid. It intersects the $xy$-plane at the level $z = 0$ which is equivalent to the circle of equation $x^2 + y^2 = 4$.

In polar coordinates, we have

$$R: \ 0 \leq \theta \leq 2\pi, \ \ 0 \leq r \leq 2.$$ 

Therefore,

$$V = \iint_R z \, dA$$

$$= \iint_R (4 - x^2 - y^2) \, dA$$

$$= \int_0^{2\pi} \int_0^2 (4 - r^2) \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta$$

$$= \left. \int_0^{2\pi} \left( 2r^2 - \frac{r^4}{4} \right) \, dr \right|_0^2$$

$$= \int_0^{2\pi} 4 \, d\theta$$

$$= 8\pi$$

Example 4. Evaluate the following double integral.

$$\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx$$

Solution. We have a double integral over a region $R$ described in rectangular coordinates as

$$R: \ 0 \leq x \leq \sqrt{2}, \ \ x \leq y \leq \sqrt{4 - x^2}.$$
Evaluating this double integral is much easier if we switch to polar coordinates. In polar coordinates, the line $y = x$ corresponds to $\theta = \pi/4$ (note that $\pi/4 = 45^\circ$). Therefore, region $R$ can be described as

$$R: \quad \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 2.$$ 

Therefore,

$$\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx = \int_{\pi/4}^{\pi/2} \int_0^2 (r^2) \, r \, dr \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \int_0^2 r^3 \, dr \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \frac{r^4}{4} \bigg|_0^2 \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} 4 \, d\theta$$

$$= 4 \left( \frac{\pi}{2} - \frac{\pi}{4} \right)$$

$$= \pi$$