Centroids

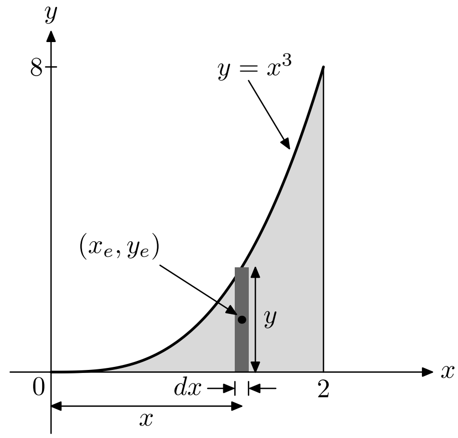
Consider a region in the plane of area A. We can think of the region as a thin plate with uniform thickness and density. The **centroid** of the region has coordinates $(\overline{x}, \overline{y})$. It can be found using

$$\overline{x} = \frac{1}{A}\int_A x_e\,dA \quad \text{and} \quad \overline{y} = \frac{1}{A}\int_A y_e\,dA$$

where (x_e, y_e) is the coordinates of the centroid of the differential element of area dA.

Example 1. Find the centroid of the region bounded by the curve $y = x^3$ and the lines $y = 0$ and $x = 2$.

Solution. We will use differential elements consisting of rectangular vertical slices of height y and width dx. This means that variable x will be the variable of integration.



We first find the area A.

$$A = \int_A dA = \int_0^2 \underbrace{y\,dx}_{dA} = \int_0^2 x^3\,dx = \left.\frac{x^4}{4}\right|_0^2 = \frac{2^4}{4} = 4$$

Now, observe that the centroid of the differential element has coordinates (x_e, y_e) where

$$x_e = x \quad \text{and} \quad y_e = \frac{y}{2}.$$

Therefore,

$$\int_A x_e\,dA = \int_0^2 x\,\underbrace{y\,dx}_{dA} = \int_0^2 x(x^3)\,dx = \int_0^2 x^4\,dx = \left.\frac{x^5}{5}\right|_0^2 = \frac{2^5}{5} = \frac{32}{5}$$

and

$$\int_A y_e\,dA = \int_0^2 \frac{y}{2}\,\underbrace{y\,dx}_{dA} = \int_0^2 \frac{y^2}{2}\,dx = \int_0^2 \frac{(x^3)^2}{2}\,dx = \int_0^2 \frac{x^6}{2}\,dx = \left.\frac{x^7}{14}\right|_0^2 = \frac{2^7}{14} = \frac{64}{7}.$$

We can now find the coordinates of the centroid.

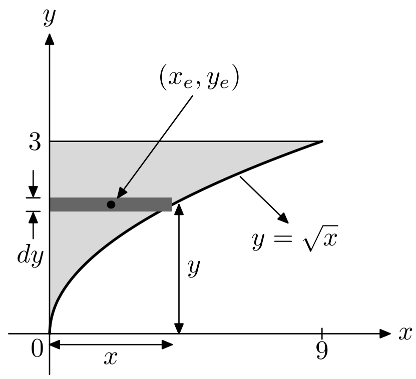
$$\overline{x} = \frac{1}{A}\int_A x_e\,dA = \frac{32/5}{4} = \frac{8}{5} \qquad \text{and} \qquad \overline{y} = \frac{1}{A}\int_A y_e\,dA = \frac{64/7}{4} = \frac{16}{7}$$

We conclude that the centroid is located at the following point.

$$\boxed{(\overline{x}, \overline{y}) = \left(\frac{8}{5}, \frac{16}{7}\right)}$$

Example 2. Find the centroid of the region bounded by the curve $y = \sqrt{x}$ and the lines $x = 0$ and $y = 3$.

Solution. We will use differential elements consisting of rectangular horizontal slices of height x and width dy. This means that variable y will be the variable of integration.



Observe that

$$y = \sqrt{x} \quad \text{implies that} \quad x = y^2.$$

Let's find the area A.

$$A = \int_A dA = \int_0^3 \underbrace{x\,dy}_{dA} = \int_0^3 y^2\,dy = \left.\frac{y^3}{3}\right|_0^3 = \frac{3^3}{3} = 9$$

Now, observe that the centroid of the differential element has coordinates (x_e, y_e) where

$$x_e = \frac{x}{2} \quad \text{and} \quad y_e = y.$$

Therefore,

$$\int_A x_e\,dA = \int_0^3 \frac{x}{2}\underbrace{x\,dy}_{dA} = \int_0^3 \frac{x^2}{2}\,dy = \int_0^3 \frac{(y^2)^2}{2}\,dy = \int_0^3 \frac{y^4}{2}\,dy = \left.\frac{y^5}{10}\right|_0^3 = \frac{3^5}{10} = \frac{243}{10}$$

and

$$\int_A y_e\,dA = \int_0^3 y\underbrace{x\,dy}_{dA} = \int_0^3 y(y^2)\,dy = \int_0^3 y^3\,dy = \left.\frac{y^4}{4}\right|_0^3 = \frac{3^4}{4} = \frac{81}{4}.$$

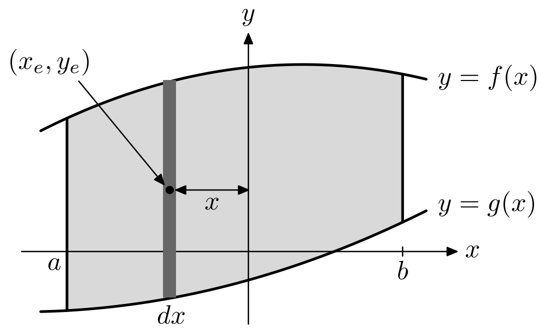
We can now find the coordinates of the centroid.

$$\overline{x} = \frac{1}{A}\int_A x_e\,dA = \frac{243/10}{9} = \frac{27}{10} \qquad \text{and} \qquad \overline{y} = \frac{1}{A}\int_A y_e\,dA = \frac{81/4}{9} = \frac{9}{4}$$

We conclude that the centroid is located at the following point.

$$\boxed{(\overline{x}, \overline{y}) = \left(\frac{27}{10}, \frac{9}{4}\right)}$$

Let's now consider the area between two curves. First consider the area bounded by the curves $y = f(x)$ and $y = g(x)$ over $a \le x \le b$.



In this case, we have

$$x_e = x \quad \text{and} \quad y_e = \frac{f(x) + g(x)}{2}.$$

Therefore,

$$A = \int_A dA = \int_a^b \underbrace{(f(x) - g(x))\, dx}_{dA}$$

$$\int_A x_e\, dA = \int_a^b x \underbrace{(f(x) - g(x))\, dx}_{dA} \quad \text{and} \quad \int_A y_e\, dA = \int_a^b \left(\frac{f(x) + g(x)}{2}\right) \underbrace{(f(x) - g(x))\, dx}_{dA}.$$

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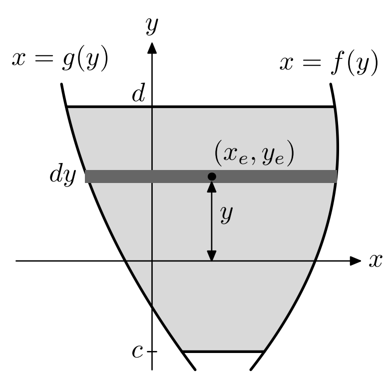
Now consider the area bounded by the curves $x = f(y)$ and $x = g(y)$ over $c \le y \le d$.



In this case, we have

$$x_e = \frac{f(y) + g(y)}{2} \quad \text{and} \quad y_e = y.$$

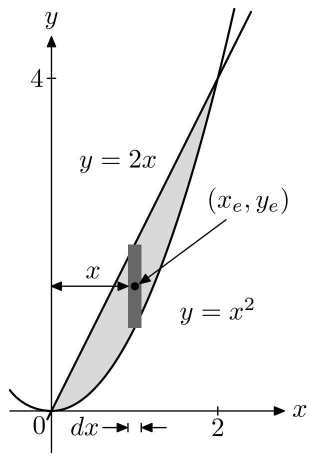
Therefore,

$$A = \int_A dA = \int_c^d \underbrace{(f(y) - g(y))\, dy}_{dA}$$

$$\int_A x_e\, dA = \int_c^d \left(\frac{f(y) + g(y)}{2}\right) \underbrace{(f(y) - g(y))\, dy}_{dA} \quad \text{and} \quad \int_A y_e\, dA = \int_c^d y \underbrace{(f(y) - g(y))\, dy}_{dA}$$

Example 3. Find the centroid of the region between the curves $y = 2x$ and $y = x^2$ over the interval $0 \le x \le 2$.

Solution. We will use differential elements consisting of rectangular vertical slices of width dx. This means that variable x will be the variable of integration.



Let's find the area A.

$$A = \int_A dA = \int_0^2 \underbrace{(2x - x^2)\,dx}_{dA} = \left(x^2 - \frac{x^3}{3}\right)\Bigg|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

Now, observe that

$$x_e = x \quad \text{and} \quad y_e = \frac{2x + x^2}{2}.$$

Therefore,

$$\int_A x_e\,dA = \int_0^2 x\,\underbrace{(2x - x^2)\,dx}_{dA} = \int_0^2 (2x^2 - x^3)\,dx = \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^2 = \frac{16}{3} - 4 = \frac{4}{3}$$

and

$$\int_A y_e\,dA = \int_0^2 \left(\frac{2x + x^2}{2}\right) \underbrace{(2x - x^2)\,dx}_{dA} = \int_0^2 \frac{4x^2 - x^4}{2}\,dx = \left(\frac{2x^3}{3} - \frac{x^5}{10}\right)\Bigg|_0^2 = \frac{16}{3} - \frac{16}{5} = \frac{32}{15}.$$

We can now find the coordinates of the centroid.

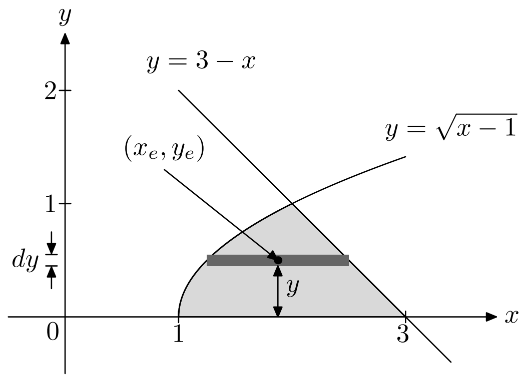
$$\overline{x} = \frac{1}{A}\int_A x_e\,dA = \frac{4/3}{4/3} = 1 \qquad \text{and} \qquad \overline{y} = \frac{1}{A}\int_A y_e\,dA = \frac{32/15}{4/3} = \frac{8}{5}$$

We conclude that the centroid is located at the following point.

$$\boxed{(\overline{x}, \overline{y}) = \left(1, \frac{8}{5}\right)}$$

Example 4. Find the centroid of the region bounded by the curves $y = 3 - x$, $y = \sqrt{x-1}$, and $y = 0$.

Solution. We will use differential elements consisting of horizontal vertical slices of width dy. This means that variable y will be the variable of integration.



Let's find the point of intersection of $y = 3 - x$ and $y = \sqrt{x-1}$.

$$3 - x = \sqrt{x-1} \implies 9 - 6x + x^2 = x - 1 \implies x^2 - 7x + 10 = 0 \implies (x-2)(x-5) = 0$$

We conclude that $x = 2$ is the solution. Then, the point of intersection is $(2, 1)$. Now, observe that

$$y = \sqrt{x-1} \implies x = y^2 + 1 \quad \text{and} \quad y = 3 - x \implies x = 3 - y.$$

We can describe the region as the region between the curves $x = y^2 + 1$ and $x = 3 - y$ over $0 \le y \le 1$.

Let's find the area A.

$$A = \int_A dA = \int_0^1 \underbrace{\left[(3-y) - (y^2+1)\right] dy}_{dA} = \int_0^1 (2 - y - y^2)\, dy = \left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 = \frac{7}{6}$$

Now, observe that

$$x_e = \frac{(3-y) + (y^2+1)}{2} \quad \text{and} \quad y_e = y.$$

Therefore,

$$\int_A x_e\, dA = \int_0^1 \left[\frac{(3-y) + (y^2+1)}{2}\right] \underbrace{\left[(3-y) - (y^2+1)\right] dy}_{dA} = \int_0^1 \frac{(3-y)^2 - (y^2+1)^2}{2}\, dy$$

$$= \int_0^1 \frac{8 - 6y - y^2 - y^4}{2}\, dy = \left(4y - \frac{3y^2}{2} - \frac{y^3}{6} - \frac{y^5}{10}\right)\Big|_0^1 = \frac{67}{30}$$

and

$$\int_A y_e\, dA = \int_0^1 y \underbrace{\left[(3-y) - (y^2+1)\right] dy}_{dA} \int_0^1 (2y - y^2 - y^3)\, dy = \left(y^2 - \frac{y^3}{3} - \frac{y^4}{4}\right)\Big|_0^1 = \frac{5}{12}$$

We can now find the coordinates of the centroid.

$$\overline{x} = \frac{1}{A}\int_A x_e\, dA = \frac{67/30}{7/6} = \frac{67}{35} \qquad \text{and} \qquad \overline{y} = \frac{1}{A}\int_A y_e\, dA = \frac{5/12}{7/6} = \frac{5}{14}$$

We conclude that the centroid is located at the following point.

$$\boxed{(\overline{x}, \overline{y}) = \left(\frac{67}{35}, \frac{5}{14}\right)}$$