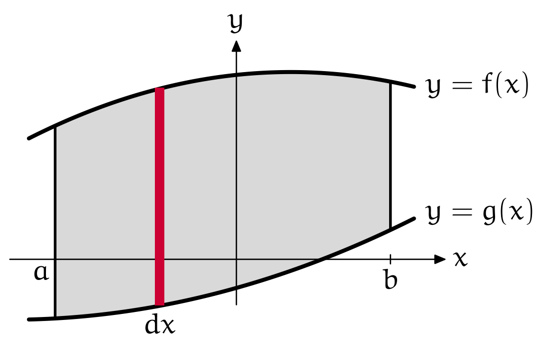
Area between curves

Vertical slices.
The area between the curves $y = f(x)$ and $y = g(x)$ over $a \leqslant x \leqslant b$ is obtained as follows.

$$\boxed{A = \int_a^b (f(x) - g(x))\,dx}$$

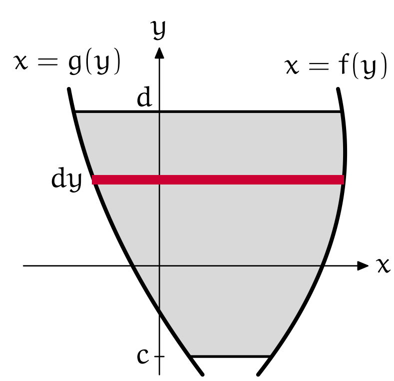
Observe that the rectangular slice has length given by $f(x) - g(x)$ and width dx.



Horizontal slices.
The area between the curves $x = f(y)$ and $x = g(y)$ over $c \leqslant y \leqslant d$ is obtained as follows.

$$\boxed{A = \int_c^d (f(y) - g(y))\,dy}$$

Observe that the rectangular slice has length given by $f(y) - g(y)$ and width dy.

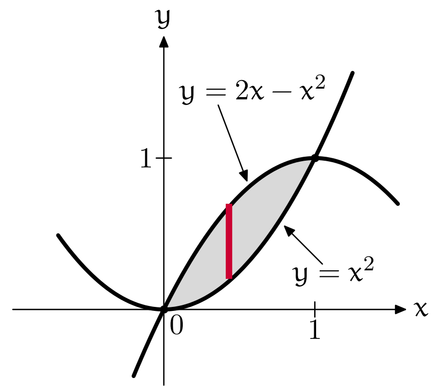


Example 1. Find the area bounded by the two parabolas $y = x^2$ and $y = 2x - x^2$.

Solution. We first find the points of intersection of the parabolas by solving

$$x^2 = 2x - x^2 \iff 2x^2 - 2x = 0 \iff 2x(x-1) = 0.$$

The solutions are $x = 0$ or 1. The points of intersections are then $(0,0)$ and $(1,1)$.

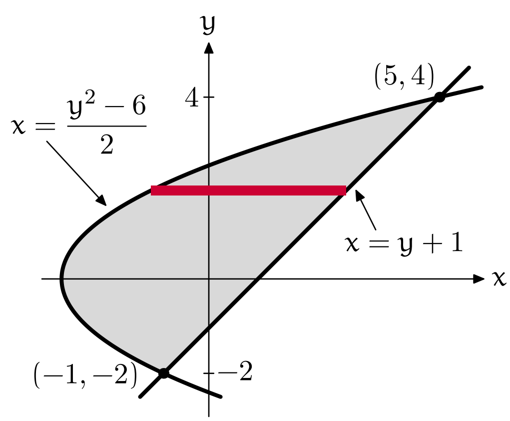


We find the area using vertical slices as follows.

$$\begin{aligned} A &= \int_0^1 (2x - x^2 - x^2)\,dx = \int_0^1 (2x - 2x^2)\,dx \\ &= \left(x^2 - \frac{2x^3}{3}\right)\Big|_0^1 \\ &= 1 - \frac{2}{3} = \boxed{\frac{1}{3}} \end{aligned}$$

Example 2. Find the area between the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution. By solving $y + 1 = (y^2 - 6)/2$, we can deduce that the points of intersection are $(-1,-2)$ and $(5,4)$.



We find the area using horizontal slices as follows.

$$\begin{aligned} A &= \int_{-2}^{4} \left[(y+1) - \left(\frac{y^2-6}{2}\right)\right] dy \\ &= \int_{-2}^{4} \left(-\tfrac{1}{2}y^2 + y + 4\right) dy = \left[-\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y\right]\Big|_{-2}^{4} \\ &= -\tfrac{1}{6}(64) + 8 + 16 - \left(\tfrac{8}{6} + 2 - 8\right) = \boxed{18} \end{aligned}$$