Area between curves

Vertical slices.
The area between the curves $y = f(x)$ and $y = g(x)$ over $a \leq x \leq b$ is obtained as follows.

$$A = \int_{a}^{b} (f(x) - g(x)) \, dx$$

Observe that the rectangular slice has length given by $f(x) - g(x)$ and width $dx$.

Horizontal slices.
The area between the curves $x = f(y)$ and $x = g(y)$ over $c \leq y \leq d$ is obtained as follows.

$$A = \int_{c}^{d} (f(y) - g(y)) \, dy$$

Observe that the rectangular slice has length given by $f(y) - g(y)$ and width $dy$. 
Example 1. Find the area bounded by the two parabolas \( y = x^2 \) and \( y = 2x - x^2 \).

Solution. We first find the points of intersection of the parabolas by solving
\[
x^2 = 2x - x^2 \iff 2x^2 - 2x = 0 \iff 2x(x - 1) = 0.
\]
The solutions are \( x = 0 \) or \( 1 \). The points of intersections are then \((0, 0)\) and \((1, 1)\).

We find the area using vertical slices as follows.
\[
A = \int_{0}^{1} (2x - x^2 - x^2) \, dx = \int_{0}^{1} (2x - 2x^2) \, dx
\]
\[
= \left[ x^2 - \frac{2x^3}{3} \right]_{0}^{1}
\]
\[
= 1 - \frac{2}{3} = \frac{1}{3}
\]

Example 2. Find the area between the line \( y = x - 1 \) and the parabola \( y^2 = 2x + 6 \).

Solution. By solving \( y + 1 = (y^2 - 6)/2 \), we can deduce that the points of intersection are \((-1, -2)\) and \((5, 4)\).

We find the area using horizontal slices as follows.
\[
A = \int_{-2}^{4} \left[ (y + 1) - \left( \frac{y^2 - 6}{2} \right) \right] \, dy
\]
\[
= \int_{-2}^{4} \left( -\frac{1}{2}y^2 + y + 4 \right) \, dy = \left[ -\frac{1}{2} \left( \frac{y^3}{3} \right) + \frac{y^2}{2} + 4y \right]_{-2}^{4}
\]
\[
= -\frac{1}{6}(64) + 8 + 16 - \left( \frac{8}{6} + 2 - 8 \right) = 18
\]