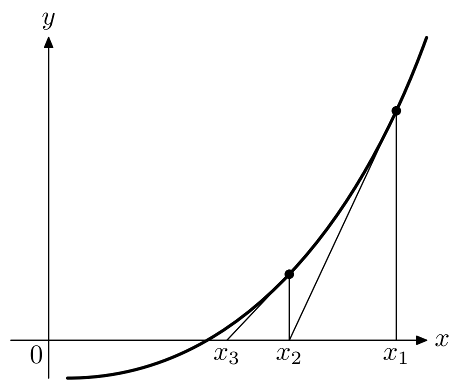
Newton's Method

Newton's method is an iterative method to approximate the roots of a function. To approximate a root r of a differentiable function f, we use the following steps.

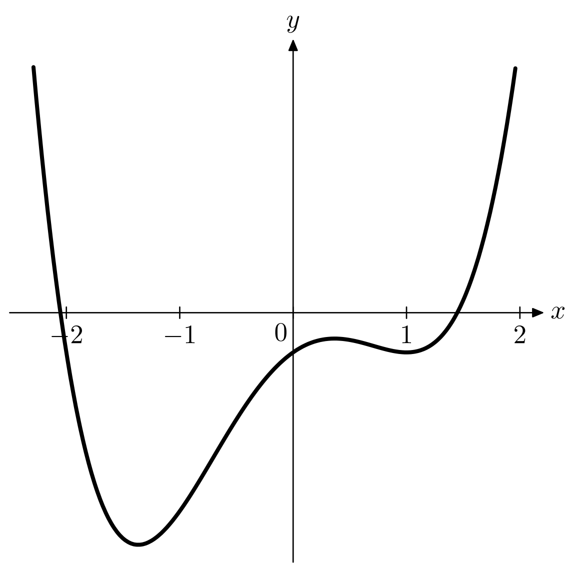
1. Make an initial estimate x_1, that is "close" to r. (A graph can be useful.)
2. Compute a new approximation using
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$
3. If $|x_{n+1} - x_n|$ is small enough, we stop and use x_{n+1} as our approximation. Otherwise, go back to step 2 and compute a new approximation.



For example let's approximate the real roots of the following function.

$$f(x) = x^4 - 3x^2 + 2x - 1$$

The graph shows that there are two real roots.



Since $f'(x) = 4x^3 - 6x + 2$, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 3x_n^2 + 2x_n - 1}{4x_n^3 - 6x_n + 2}$$

Let's first approximate the positive root. From the graph we see that it is about halfway between 1 and 2 so let's choose $x_1 = 1.5$ as our first estimate.

n	x_n	$f(x_n)$
1	1.5	0.3125
2	1.451 923 077	0.023 608 195
3	1.447 655 156	0.000 175 297
4	1.447 622 989	9.9×10^{-9}
5	1.447 622 987	

We conclude that the positive root is 1.447 622 98, accurate to eight decimal places.

Now let's proceed in the same way to estimate the negative root. From the graph we see that it is close to -2 so let's choose $x_1 = -2$ as our first estimate.

n	x_n	$f(x_n)$
1	-2	-1
2	$-2.055\,555\,556$	0.066 196 083
3	$-2.052\,311\,936$	0.000 234 885
4	$-2.052\,300\,345$	2.99×10^{-9}
5	$-2.052\,300\,345$	

We conclude that the negative root is $-2.052\,300\,345$, accurate to nine decimal places.

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To make an initial estimate of a root of a continuous function, we can use the following idea. If the numbers $f(a)$ and $f(b)$ have different signs, then there is a root inside the interval $[a, b]$. We can use the middle point of the interval as our initial estimate.

For example consider the function $f(x) = x^5 + x - 1$. We see that

$$f(0) = -1 < 0 \quad \text{and} \quad f(1) = 1 > 0.$$

We can conclude that there is a root in the interval $[0, 1]$. We could choose $x_1 = 0.5$ as our first estimate.