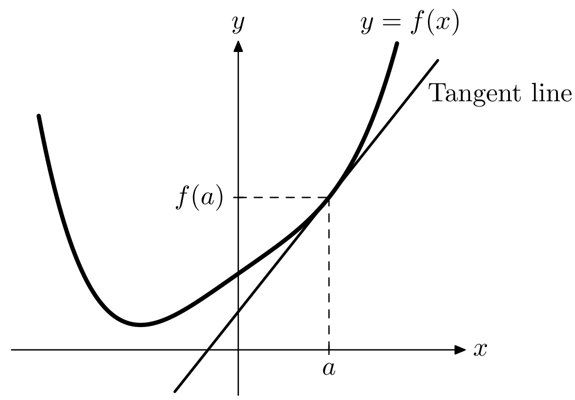
Linear Approximations

Consider a differentiable function f. The tangent line at point $(a, f(a))$ has a slope equal to $f'(a)$. An equation for the tangent line is

$$y = f(a) + f'(a)(x - a).$$



The tangent line can be used to approximate $f(x)$ for values of x close to a.

$$\boxed{f(x) \approx f(a) + f'(a)(x - a), \quad \text{for } x \text{ close to } a}$$

This is called the **linear approximation** of f near $x = a$.

Example. Find the equation of the tangent line of $f(x) = \sqrt{x}$ at $x = 9$. Use a linear approximation to estimate $\sqrt{9.2}$.

Solution. We have $f'(x) = \dfrac{1}{2\sqrt{x}}$, then

$$f(9) = \sqrt{9} = 3 \quad \text{and} \quad f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}.$$

The tangent line has a slope $m = 1/6$ and passes through point $(9, 3)$. The equation of the tangent line is

$$y = 3 + \frac{1}{6}(x - 9).$$

For values of x close to 9, we have

$$\sqrt{x} \approx 3 + \frac{1}{6}(x - 9).$$

If we set $x = 9.2$, we obtain

$$\sqrt{9.2} \approx 3 + \frac{1}{6}(9.2 - 9) \approx 3.033.$$

We can use a calculator to confirm that this is a very good estimate of $\sqrt{9.2}$.

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