Linear Approximations

Consider a differentiable function $f$. The tangent line at point $(a, f(a))$ has a slope equal to $f'(a)$. An equation for the tangent line is

$$y = f(a) + f'(a)(x - a).$$

The tangent line can be used to approximate $f(x)$ for values of $x$ close to $a$.

$$f(x) \approx f(a) + f'(a)(x - a), \text{ for } x \text{ close to } a$$

This is called the linear approximation of $f$ near $x = a$.

**Example.** Find the equation of the tangent line of $f(x) = \sqrt{x}$ at $x = 9$. Use a linear approximation to estimate $\sqrt{9.2}$.

**Solution.** We have $f'(x) = \frac{1}{2\sqrt{x}}$, then

$$f(9) = \sqrt{9} = 3 \quad \text{and} \quad f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}.$$ 

The tangent line has a slope $m = 1/6$ and passes through point $(9, 3)$. The equation of the tangent line is

$$y = 3 + \frac{1}{6}(x - 9).$$

For values of $x$ close to 9, we have

$$\sqrt{x} \approx 3 + \frac{1}{6}(x - 9).$$

If we set $x = 9.2$, we obtain

$$\sqrt{9.2} \approx 3 + \frac{1}{6}(9.2 - 9) \approx 3.033.$$ 

We can use a calculator to confirm that this is a very good estimate of $\sqrt{9.2}$. 

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