Curve sketching examples

Example 1. Use derivatives to sketch the graph of $y = x^3 - 3x^2 + 5$.

Solution: The first two derivatives are

$$y' = 3x^2 - 6x = 3x(x-2)$$
$$y'' = 6x - 6 = 6(x-1).$$

We have $y' = 0$ for $x = 0$ and $x = 2$. We say that $x = 0$ and $x = 2$ are critical numbers. Let's make a sign chart to analyze the signs of y'.

	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$3x$	$-$	$+$	$+$
$(x-2)$	$-$	$-$	$+$
y'	$+$	$-$	$+$
y	$\nearrow$	$\searrow$	$\nearrow$

Observe that $x = 0$ corresponds to a relative maximum and $x = 2$ corresponds to a relative minimum.

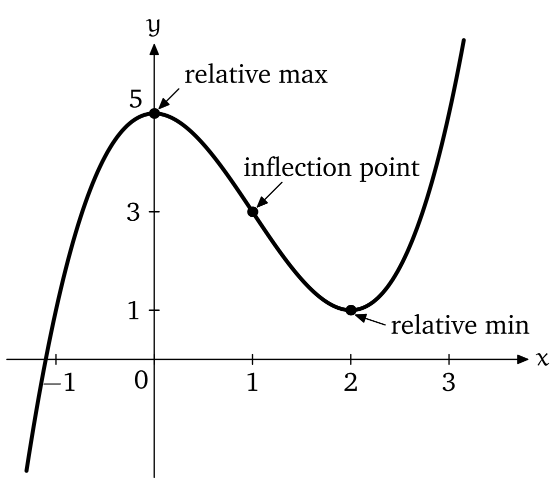
We have $y'' = 0$ for $x = 1$. Let's make a sign chart to analyse the signs of y''.

	$(-\infty, 1)$	$(1, \infty)$
$(x-1)$	$-$	$+$
y''	$-$	$+$
y	$\cap$	$\cup$

Observe that concavity changes when $x = 1$. This means that when $x = 1$, we have an inflection point.

Let's now summarize all our information and sketch the graph.

	y	y'	y''	Characteristic of Graph
$-\infty < x < 0$		$+$	$-$	Increasing $\nearrow$, concave down $\cap$
$x = 0$	5	0	$-$	Relative maximum
$0 < x < 1$		$-$	$-$	Decreasing $\searrow$, concave down $\cap$
$x = 1$	3	$-$	0	Inflection point
$1 < x < 2$		$-$	$+$	Decreasing $\searrow$, concave up $\cup$
$x = 2$	1	0	$+$	Relative minimum
$2 < x < \infty$		$+$	$+$	Increasing $\nearrow$, concave up $\cup$



To find the x-intercept, we can use Newton's method to get $x \approx -1.1038$.

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Example 2. Use derivatives to sketch the graph of $y = \dfrac{x^2+5x+1}{x^2} = 1 + \dfrac{5}{x} + \dfrac{1}{x^2}$.

Solution: Since $x = 0$ makes the denominator of y equal to 0, we conclude that $x = 0$ corresponds to a vertical asymptote.

As $x \to \infty$ and $x \to -\infty$, we have $y \to 1$. Therefore, $y = 1$ corresponds to a horizontal asymptote.

To find the x—intercepts, we solve $x^2+5x+1 = 0$. Using the quadratic formula, we find two x—intercepts.

$$x = \frac{-5 \pm \sqrt{21}}{2} \implies x \approx -4.791 \text{ or } x \approx -0.209$$

The first two derivatives are

$$y' = -\frac{5}{x^2} - \frac{2}{x^3} = \frac{-(5x+2)}{x^3} \quad \text{and} \quad y'' = \frac{10}{x^3} + \frac{6}{x^4} = \frac{10x+6}{x^4}.$$

We have $y' = 0$ for $x = -2/5$ and y' is undefined for $x = 0$. Let's make a sign chart for y'.

	$(-\infty, -2/5)$	$(-2/5, 0)$	$(0, \infty)$
$-(5x+2)$	$+$	$-$	$-$
x^3	$-$	$-$	$+$
y'	$-$	$+$	$-$
y	$\searrow$	$\nearrow$	$\searrow$

We have $y'' = 0$ for $x = -3/5$ and y'' is undefined for $x = 0$.. Let's make a sign chart for y''.

	$(-\infty, -3/5)$	$(-3/5, 0)$	$(0, \infty)$
$10x+6$	$-$	$+$	$+$
x^4	$+$	$+$	$+$
y''	$-$	$+$	$+$
y	$\cap$	$\cup$	$\cup$

Let's now summarize all our information and sketch the graph.

	y	y'	y''	Characteristic of Graph
$-\infty < x < -3/5$		$-$	$-$	Decreasing $\searrow$, concave down $\cap$
$x = -3/5$	$-41/9$	$-$	0	Inflection point
$-3/5 < x < -2/5$		$-$	$+$	Decreasing $\searrow$, concave up $\cup$
$x = -2/5$	$-21/4$	0	$+$	Relative minimum
$-2/5 < x < 0$		$+$	$+$	Increasing $\nearrow$, concave up $\cup$
$x = 0$	Undef.	Undef.	Undef.	Vertical asymptote
$0 < x < \infty$		$-$	$+$	Decreasing $\searrow$, concave up $\cup$