Basics of Complex Numbers

Negative numbers do not have a square root that can be expressed by real numbers. To overcome this, an *imaginary* number $i$ has been defined as the square root of minus one.

$$i = \sqrt{-1}$$

The square root of any negative number can be expressed in terms of $i$. For example

$$\sqrt{-9} = \sqrt{9} \sqrt{-1} = 3i.$$  

A **complex number** is a number of the form

$$z = a + bi$$

where $a$ and $b$ are real numbers called the real and imaginary parts of $z$.

$$\text{Re}(z) = a \quad \text{and} \quad \text{Im}(z) = b$$

**Example.** Let’s find the roots of the quadratic equation

$$x^2 - 6x + 25 = 0.$$  

**Solution.** We can use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to find the roots.

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(25)}}{2(1)} = \frac{6 \pm \sqrt{-64}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i.$$  

There are two complex roots: $3 + 4i$ and $3 - 4i$.

Graphically, a complex number $z = a + bi$ can be represented in a rectangular coordinate system called the complex plane. The $x$-axis is called the real-axis and the $y$-axis is called the imaginary-axis.

![Diagram of complex plane](image)

By drawing a vector from the origin to the point $z = a + bi$, an angle $\theta$ in standard position is formed. The magnitude of the vector is $r$. Observe that

$$a = r \cos \theta \quad \text{and} \quad b = r \sin \theta.$$  

We then have

$$z = a + bi = r(\cos \theta + i \sin \theta).$$  

The polar form of the complex number is $z = r(\cos \theta + i \sin \theta)$. Thanks to Euler’s formula

$$e^{i\theta} = \cos \theta + i \sin \theta$$

we can also represent a complex number in exponential form as

$$z = re^{i\theta}.$$