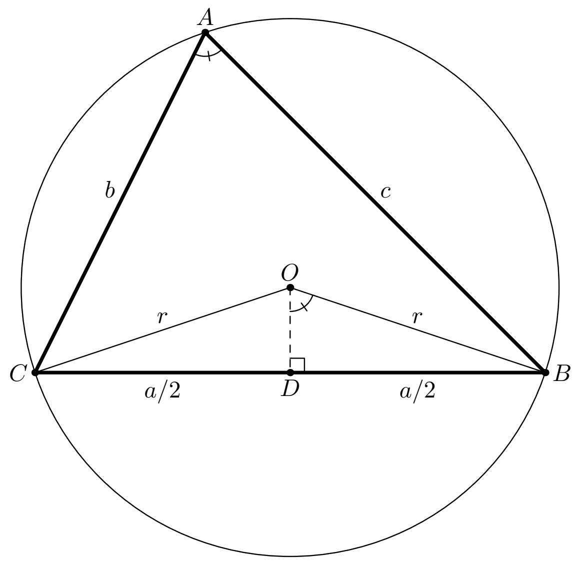
Proof of the Law of Sines and the Law of Cosines

Law of Sines.
Consider a triangle ABC inscribed in a circle with center O and radius r.



From basic geometry, we know that $A = \angle CAB = \angle DOB$. From the right triangle DOB, we deduce

$$\sin A = \frac{a/2}{r} \implies \frac{a}{\sin A} = 2r.$$

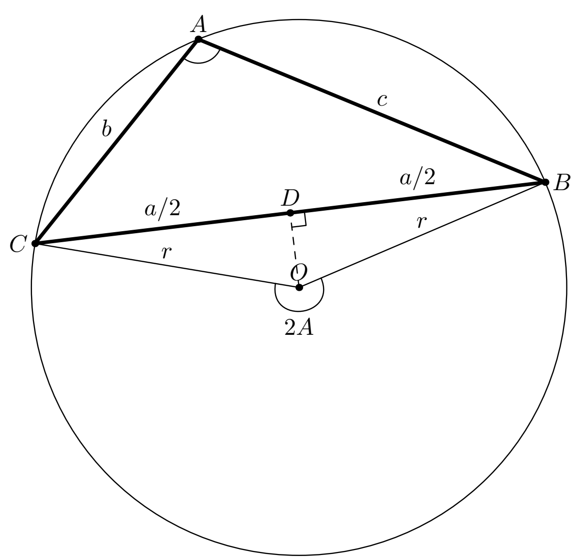
By repeating the same argument for angles B and C, we obtain the Law of Sines.

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r}$$

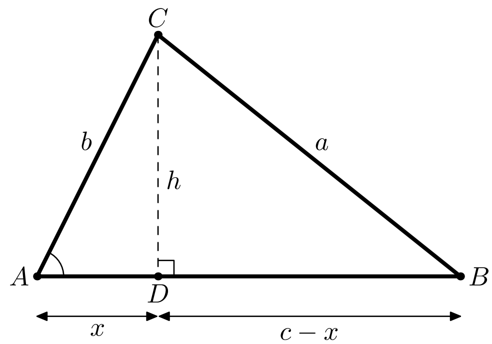
Observe that the proof is similar if the center O is not inside triangle ABC. Since the outside angle $\angle COB$ satisfies $\angle COB = 2A$, we have $\angle DOB = 180^\circ - A$. Since

$$\sin(180^\circ - A) = \sin A,$$

we have $\sin A = \dfrac{a/2}{r} \implies \dfrac{a}{\sin A} = 2r$.



Law of Cosines.
Consider the following triangle ABC.



From the right triangle ADC, we deduce

$$x^2 + h^2 = b^2 \tag{1}$$

and

$$\cos A = \frac{x}{b} \implies x = b\cos A. \tag{2}$$

From the right triangle BDC, we deduce

$$(c - x)^2 + h^2 = a^2 \implies a^2 = c^2 - 2cx + (x^2 + h^2). \tag{3}$$

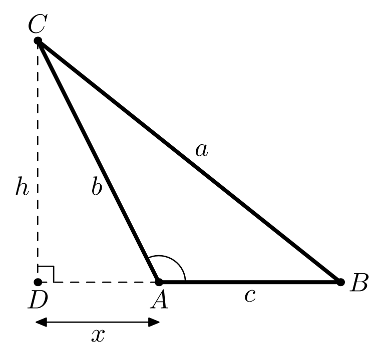
Substituting equations (1) and (2) into (3), we get

$$a^2 = c^2 - 2c(b\cos A) + b^2$$

which is the Law of Cosines.

$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$$

The proof is similar if angle A is obtuse.



From the right triangle ADC, we deduce

$$x^2 + h^2 = b^2 \tag{4}$$

and

$$\cos(180^\circ - A) = -\cos A = \frac{x}{b} \implies x = -b\cos A. \tag{5}$$

From the right triangle BDC, we deduce

$$(c + x)^2 + h^2 = a^2 \implies a^2 = c^2 + 2cx + (x^2 + h^2). \tag{6}$$

Substituting equations (4) and (5) into (6) gives the Law of Cosines.