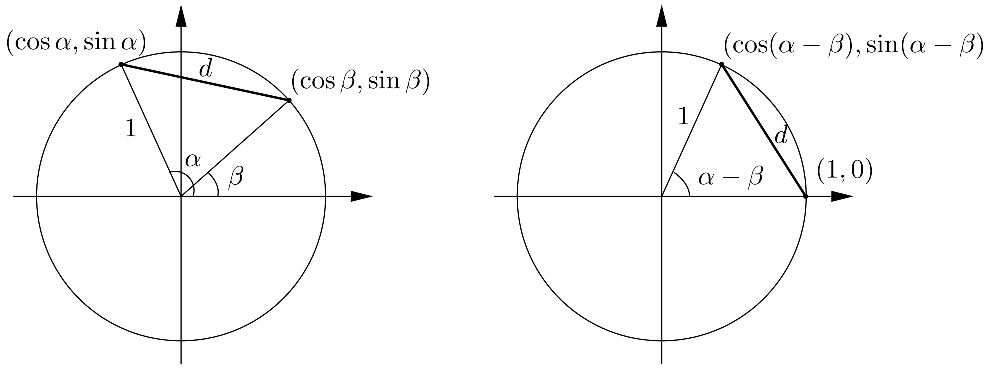
Proof of Sum and Difference Identities

We will prove the following trigonometric identities.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Proof. Consider two angles α and β. The distance d in the following two unit circles are equal.



From the first one we obtain
$$d = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}.$$
From the second one we obtain
$$d = \sqrt{(\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta) - 0)^2}.$$
From these two expressions for d, we can deduce
$$\begin{aligned} d^2 &= d^2 \\ (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 &= (\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta) - 0)^2 \\ \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta &= \cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta) \\ (\cos^2\alpha + \sin^2\alpha) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) + (\cos^2\beta + \sin^2\beta) &= (\cos^2(\alpha-\beta) + \sin^2(\alpha-\beta)) - 2\cos(\alpha-\beta) + 1 \\ 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) &= 2 - 2\cos(\alpha-\beta). \end{aligned}$$
Therefore,
$$\boxed{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.}$$
Replacing β by $-\beta$ gives us
$$\cos(\alpha-(-\beta)) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$
Then,
$$\boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.}$$
Now let's replace α by $\frac{\pi}{2} - \alpha$ to get
$$\cos(\tfrac{\pi}{2} - \alpha + \beta) = \cos(\tfrac{\pi}{2} - \alpha)\cos\beta - \sin(\tfrac{\pi}{2} - \alpha)\sin\beta.$$
Since we know that
$$\sin(\tfrac{\pi}{2} - \alpha) = \cos\alpha, \quad \cos(\tfrac{\pi}{2} - \alpha) = \sin\alpha, \quad \text{and} \quad \cos(\tfrac{\pi}{2} - (\alpha-\beta)) = \sin(\alpha-\beta)$$
we can conclude that
$$\boxed{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.}$$
Finally, by replacing β by $-\beta$ we obtain
$$\sin(\alpha-(-\beta)) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$
Then,
$$\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.}$$

□

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